
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magical transformations, do not "box" your final results. Show me all the magic on the page, for I do not read minds.*

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1. (5 pts.) State the Fundamental Theorem of Calculus.

Suppose that f is continuous on the closed interval [a,b].

Part 1: If the function g is defined on [a,b] by

$$g(x) = \int_a^x f(t)\,dt,$$

then g is an antiderivative of f. That is, $g'(x) = f(x)$ for each x in [a,b]. [This, really, is the punch line!!]

Part 2: If G is any antiderivative of f on [a,b], then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

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2. (5 pts.) Complete the equation to write the following sum using sigma notation:

$$\frac{x^2}{1} - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots - \frac{x^{44}}{22} = \sum_{i=1}^{22} \frac{(-1)^{i+1}x^{2i}}{i}$$

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3. (10 pts.) Evaluate each of the following sums in closed form.

(a)

$$\sum_{i=1}^{20}(i^2+1) = \sum_{i=1}^{20} i^2 + \sum_{i=1}^{20} 1$$

$$= \frac{(20)(21)(41)}{6} + 20$$

$$= (10)(7)(41) + 20 = (70)(41) + 20 = 2890$$

(b) $$\sum_{i=0}^{99}\left(\frac{1}{5}\right)^i = \frac{1-(1/5)^{99+1}}{1-(1/5)} = \frac{5}{4}(1-(1/5)^{100})$$

4. (5 pts.) Using appropriate properties of the definite integral and suitable area formulas from geometry, evaluate the following definite integral. [Roughly sketching a couple of simple graphs might help. First use linearity, though.]

$$\int_{-2}^{2} x - (4 - x^2)^{1/2}\ dx \quad = \quad \int_{-2}^{2} x\ dx \quad - \quad \int_{-2}^{2} (4 - x^2)^{1/2}\ dx$$

$$= \quad 0 \quad - \quad 2\pi \quad = \quad -2\pi$$

since the numerical value of the first integral is oddly zero and the value of the second integral is the one half of the area of a circle with a radius of 2. ["Oddly Zero" ????]

5. (5 pts.)
Find the average value of $f(x) = \sec^2(x)$ over the interval $[-\pi/4,\pi/4]$.

$$\begin{aligned} f_{AVE} = \frac{1}{(\pi/4)-(-\pi/4)}\int_{-\pi/4}^{\pi/4} \sec^2(x)\ dx &= \frac{2}{\pi}(\tan(x))\Big|_{-\pi/4}^{\pi/4} \\ &= \frac{2}{\pi}(\tan(\pi/4)-\tan(-\pi/4)) \\ &= \frac{2}{\pi}\cdot\frac{2}{1} \\ &= \frac{4}{\pi}. \end{aligned}$$

6. (5 pts.) Express the solution to the following initial value problem by using a definite integral with respect to the variable t,

$$\frac{dy}{dx} = \sec^3(x)\ , \quad y(1) = \pi.$$

To do this, fill in the right side of the equation below correctly. Do not attempt to evaluate the definite integral.

$$y(x) = \pi + \int_1^x \sec^3(t)\ dt$$

7. (5 pts.) Give an example of a bounded function defined on the closed interval [-1,1] that is not Riemann integrable.

Here is an obnoxious old friend that does the job.

$$g(x) = \begin{cases} 0 \quad , & \text{for } x \text{ a rational number in } [-1,1] \\ 1 \quad , & \text{for } x \text{ an irrational number in } [-1,1] \end{cases}$$

5.4: Problem 55.

8. (10 pts.) Assume that $[x_{i-1},x_i]$ denotes the i*th* subinterval of a partition of the interval [0,1] into n subintervals, all with the same length $\Delta x = 1/n$.

(a) Write the value of the following limit as a definite integral.

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{\pi x_i}{4}\right)\Delta x$$

$$L \quad = \quad \int_0^1 \cos\left(\frac{\pi}{4}x\right)\,dx$$

(b) By evaluating the integral you obtained in part (a) above using the Fundamental Theorem of Calculus, give the exact numerical value of the limit, L.

$$L = \int_0^1 \cos\left(\frac{\pi}{4}x\right)\,dx = \left(\frac{4}{\pi}\sin\left(\frac{\pi}{4}x\right)\right)\Big|_0^1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{4}\right) - \sin(0)\right) = \frac{2^{3/2}}{\pi}.$$

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9. (10 pts.) Reveal all the details of evaluating the given integral by computing

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x\,,$$

where the sum is assumed to have originated from a regular partition of the given interval of integration. Here, you are to actually compute the Riemann sum in a closed form and then evaluate the limit. Reveal all the magical details.

Since $\Delta x = 2/n$, $x_i = 2i/n$ for i = 0, 1, ... , n are the end points of the intervals of the general regular partition. So

$$\begin{aligned}\int_0^2 2x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\left(\frac{2i}{n}\right)\Delta x\\ &= \lim_{n\to\infty}\left(\frac{8}{n^2}\right)\sum_{i=1}^{n} i\\ &= \lim_{n\to\infty}\left(\frac{8}{n^2}\right)\left(\frac{n(n+1)}{2}\right)\\ &= \lim_{n\to\infty} 4\left(1+\frac{1}{n}\right)\\ &= 4.\end{aligned}$$

10. (5 pts.) Using only the second comparison property of integrals, give both a lower and an upper bound on the true numerical value of the integral below.

$$I = \int_{\pi/3}^{\pi/2} \sin^2(x)\,dx$$

Let $f(x) = \sin^2(x)$. Then $f'(x) = 2\sin(x)\cos(x) > 0$ when x is in the interval $(\pi/3,\pi/2)$. So f is increasing on $[\pi/3,\pi/2]$. Thus, $3/4 = f(\pi/3) \le f(x) \le f(\pi/2) = 1$ when $\pi/3 \le x \le \pi/2$. From the 2nd comparison property of integrals, it follows that we have $\pi/8 = (3/4)(\pi/6) \le I \le (1)(\pi/6) = \pi/6$, since $\pi/2 - \pi/3 = \pi/6$.

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11. (5 pts.) A classmate showed you his solution to a certain integration problem where he made the substitution $u = \sin(x)$ so that $du = \cos(x)dx$. What he did appears below.

$$\int_0^{\pi/2} \sin^2(x)\cos(x)\,dx = \int_0^{\pi/2} u^2\,du = \left(\frac{1}{3}u^3\right)\Big|_0^{\pi/2} = \left(\frac{1}{3}\sin^3(x)\right)\Big|_0^{\pi/2} = \frac{1}{3}$$

Briefly explain to him why he will not get full credit for his work if he submits it to his friendly Math Professor. Be specific.

The first and third equations aren't true. The first equation is false as a result of a failure to apply the substitution theorem correctly by changing the limits of integration. The third equation is made false by substituting back, replacing u with sin(x). Go ahead and compute the two differences and take note of how different the results are. I've heard that lying is good in politics, but very bad in Math and Science.

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12. (10 pts.) (a) Locate the critical points of the function g defined on the interval $[-\pi,\pi]$ by means of the equation

$$g(x) = \int_0^x t^2\cos(t)\,dt.$$

[Note: The critical points are actually in the open interval $(-\pi,\pi)$.]

By applying the Fundamental Theorem of Calculus, it follows that $g'(x) = x^2\cdot\cos(x)$ for $-\pi < x < \pi$. It follows that $g'(x) = 0$ precisely where $x^2\cdot\cos(x) = 0$ in $(-\pi,\pi)$. Thus, the critical points of g are $x = -\pi/2$, $x = 0$, and $x = \pi/2$.

(b) Determine the open intervals in $(-\pi, \pi)$ where g is increasing or decreasing.

Plainly, if $x \ne 0$, the sign of $g'(x)$ is determined by the sign of $\cos(x)$ in $(-\pi,\pi)$. So $g'(x) > 0$ when $-\pi/2 < x < 0$ or $0 < x < \pi/2$, and $g'(x) < 0$ when $-\pi < x < -\pi/2$ or $\pi/2 < x < \pi$. From the continuity of g, it follows that g is increasing on the open interval $(-\pi/2,\pi/2)$, and g is decreasing on the set $(-\pi,\pi/2)\cup(\pi/2,\pi)$.

13. (15 pts.) Differentiate the following functions:
[Make sure you label your derivatives correctly.]

(a) $$g(x) = \int_x^0 \frac{1}{(4+t^2)^{1/2}}\, dt = -\int_0^x \frac{1}{(4+t^2)^{1/2}}\, dt.$$

Thus,

$$g'(x) = \frac{-1}{(4+x^2)^{1/2}}.$$

(b) $$f(x) = \int_1^{\sin(x)} \frac{1}{1+t^2}\, dt$$

By using chain rule, we have

$$f'(x) = \left[\frac{1}{1+\sin^2(x)}\right]\cdot\cos(x) = \frac{\cos(x)}{1+\sin^2(x)}.$$

(c)
$$\begin{aligned} h(x) &= \int_{x^3}^{\sin(x)} \cos(t)\, dt \\ &= \int_{x^3}^{0} \cos(t)\, dt + \int_0^{\sin(x)} \cos(t)\, dt \\ &= \int_0^{\sin(x)} \cos(t)\, dt - \int_0^{x^3} \cos(t)\, dt \end{aligned}$$

Thus,

$$h'(x) = \cos(\sin(x))\cos(x) - 3x^2\cos(x^3).$$

You can, of course, "check" your answer here since

$$h(x) = \int_{x^3}^{\sin(x)} \cos(t)\, dt = \sin(\sin(x)) - \sin(x^3).$$

14. (5 pts.) Evaluate the following definite integral.

$$\begin{aligned} \int_0^2 | x^2 - 1 |\, dx &= \int_0^1 | x^2 - 1 |\, dx + \int_1^2 | x^2 - 1 |\, dx \\ &= \int_0^1 -(x^2 - 1)\, dx + \int_1^2 x^2 - 1\, dx \\ &= (x - (x^3/3))\Big|_0^1 + ((x^3/3) - x)\Big|_1^2 \\ &= \dots = 2. \end{aligned}$$

Silly 10 Point Bonus: Reveal all the magic in evaluating the following limit. [This is easy if you grok the proof of the hard part of the F.T.of C.]

$$\lim_{x\to 0^+} \frac{1}{x}\int_{-x}^{x} \frac{\sec(t)}{\pi^2 + t^4}\, dt = \frac{2}{\pi^2}$$

See c2-t1-bo.pdf for the stinking magic.