READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magical transformations, do not "box" your final results. Show me all the magic on the page, for I do not read minds.*

Silly 10 Point Bonus: Reveal all the magic in evaluating the following limit. [This is easy if you grok the proof of the hard part of the F.T.of C.]

$$\lim_{x\to 0^+} \frac{1}{x}\int_{-x}^{x} \frac{\sec(t)}{\pi^2+t^4}\,dt =$$

I can think of three natural ways that one can see easily that

$$\lim_{x\to 0^+} \frac{1}{x}\int_{-x}^{x} \frac{\sec(t)}{\pi^2+t^4}\,dt = \frac{2}{\pi^2}.$$

First, to simplify notation a little, let's set

$$f(x) = \frac{\sec(x)}{\pi^2+x^4}.$$

Realize that f is continuous on the interval $(-\pi/2,\pi/2)$. Then the limit with which we must contend is

$$L = \lim_{x\to 0^+} \frac{1}{x}\int_{-x}^{x} f(t)\,dt$$

where f is continuous in a open interval containing 0. [By abstracting, we simplify!!]

Now the hard way to proceed is to use the hint. The key to the text's proof of the F.T.of C. is the magic of the Average Value Theorem. Using this, for each x in $(0,\pi/2)$,

$$\frac{1}{x}\int_{-x}^{x} f(t)\,dt = \frac{1}{2x}\int_{-x}^{x} 2f(t)\,dt$$

$$= 2f(\xi(x))$$

$$\textit{for some } \xi(x) \in [-x,x].$$

Note that as $x \to 0^+$, $\xi(x) \to 0$. From the continuity of $2f(x)$ at zero, then, it follows that $L = 2f(0) = 2/\pi^2$.

A second way to handle the varmint is to use the neat accident that f above is even.

$$L = \lim_{x\to 0^+} \frac{1}{x}\int_{-x}^{x} f(t)\,dt = \lim_{x\to 0^+} \frac{1}{x}\int_{0}^{x} 2f(t)\,dt$$

which may be recognized as $g'(0) = 2f(0)$ when

$$g(x) = \int_0^x 2f(t)\,dt.$$

[Rewrite the limit using h instead of x and write the upper limit of integration in the integral as "0 + h".]

Finally, a third way to unravel the mystery is to recognize that l'Hopital's Rule is applicable, and to use it. Oh, there's a catch of course. The Fundamental Theorem of Calculus should be used to differentiate the numerator. To avoid some clutter, we'll use the limit form where we have set

$$f(x) = \frac{\sec(x)}{\pi^2 + x^4}.$$

Then

$$L = \lim_{x \to 0^+} \frac{1}{x}\int_{-x}^{x} f(t)\,dt$$

$$= \lim_{x \to 0^+} \frac{\int_{-x}^{x} f(t)\,dt}{x}$$

$(L'H)$

$$= \lim_{x \to 0^+}[f(x) - f(-x)\cdot(-1)]$$

$$= 2f(0)$$

since f is continuous at zero.

So do you grok the Fundamental Theorem yet??