READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (10 pts.) Evaluate each of the following integrals. Look before you leap, for propriety may be problematical. Don't use the F.T. of C. if the integrand does not satisfy the required hypotheses.

(a)

$$\int_0^8 x^{-1/3}\,dx = \lim_{t\to 0^+}\int_t^8 x^{-1/3}\,dx = \lim_{t\to 0^+}(3/2)x^{2/3}\Big|_t^8$$

$$= \lim_{t\to 0^+}(3/2)[8^{2/3} - t^{2/3}] = 6.$$

(b)

$$\int_{-\ln(\pi)}^{\infty} e^{-x}\,dx = \lim_{t\to\infty}\int_{-\ln(\pi)}^{t} e^{-x}\,dx$$

$$= \lim_{t\to\infty}[(-e^{-x})\Big|_{-\ln(\pi)}^{t}]$$

$$= \lim_{t\to\infty}[\pi - e^{-t}] = \pi.$$

Slo-Mo: $e^{-(-\ln(\pi))} = e^{\ln(\pi)} = \pi$.

(10 pts.) Find Taylor's formula for the given function f at $a = \pi$. Find both the Taylor polynomial, $P_3(x)$, and the Lagrange form of the remainder term, $R_3(x)$, for the function $f(x) = \sin(x)$ at $a = \pi$. Then write $\sin(x)$ in terms of $P_3(x)$ and $R_3(x)$.

$$P_3(x) = \sum_{k=0}^{3}\frac{f^{(k)}(\pi)}{k!}(x-\pi)^k = \cos(\pi)(x-\pi) - \frac{\cos(\pi)}{3!}(x-\pi)^3$$

$$= -(x-\pi) + \frac{1}{6}(x-\pi)^3.$$

$$R_3(x) = \frac{f^{(4)}(c)}{4!}(x-\pi)^4 = \frac{\sin(c)}{24}(x-\pi)^4$$

for some c between x and π.

$$\sin(x) = -(x-\pi) + \frac{1}{6}(x-\pi)^3 + \frac{\sin(c)}{24}(x-\pi)^4$$

for some c between x and π.

3. (10 pts.) Determine whether the sequence $\{a_n\}$ converges, and find its limit if it does.

(a) $$a_n = \sum_{k=1}^{n}\left(\frac{3}{4}\right)^k = \sum_{j=0}^{n-1}\frac{3}{4}\left(\frac{3}{4}\right)^j = \frac{(3/4)[1-(3/4)^n]}{1-(3/4)} = 3[1-(3/4)^n].$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3[1-(3/4)^n] = 3.$$

Obviously, or not, you can also realize that this sequence is the sequence of partial sums for the convergent geomtric series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k.$$

Consequently, the limit of the sequence is the sum of the series. [Did you *grok* that definition?]

(b) $a_n = 2\pi n\cdot\sin(\frac{4}{n})$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} 2\pi n\sin(\frac{4}{n})\\ &= \lim_{n\to\infty} 8\pi\left[\frac{\sin(\frac{4}{n})}{\frac{4}{n}}\right] = 8\pi.\end{aligned}$$

[You could use L'Hopital's rule but it is really not needed.]

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4. (10 pts.) Find the radius of convergence and the interval of convergence of the power series function

$$f(x) = \sum_{k=1}^{\infty}\frac{(-1)^k(x-2)^k}{k\,10^k}\ .$$

First observe that the series is centered at x = 2. Then we apply the ratio test for absolute convergence in order to determine the radius of convergence of the power series.

$$\rho(x) = \lim_{k\to\infty}\frac{|u_{k+1}|}{|u_k|} = \ldots = \lim_{k\to\infty}\frac{1}{10}\frac{k}{k+1}|x-2| = \frac{1}{10}|x-2|.$$

Now, $\rho(x) < 1$ if, and only if $|x - 2| < 10$. Thus, R = 10 is the radius of convergence. The endpoints are $x_L = -10 + 2$ and $x_R = 2 + 10$. When you substitute x_L into the power series and simplify the algebra, you obtain $\sum_{k=0}^{\infty}\frac{1}{k+1}$ which diverges. When you substitute x_R into the power series and simplify the algebra, you obtain $\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}$ which converges. I = (-8, 12].

5. (10 pts.) Using only the integral test, determine whether the series below converges. Explicitly define the function f(x) used, and verify all the hypotheses of the theorem are true, including whether the function is decreasing.

$$\sum_{n=1}^{\infty} \frac{2}{n^2+n}$$

Here are the pieces to the puzzle:

(a) Set $f(x) = 2/(x^2 + x)$ for $x \geq 1$ since plainly $f(n) = 2/(n^2 + n)$ for $n \geq 1$.

(b) Obviously f is a positive, continuous rational function for $x \geq 1$.

(c) Since $f'(x) = -2(2x + 1)/(x^2 + x)^2$, $f'(x) < 0$ for $x > 1$. Thus, f is decreasing for $x \geq 1$.

(d) Since

$$\int_1^{\infty} \frac{2}{x^2+x}dx = \lim_{b\to\infty}\int_1^b \frac{2}{x} - \frac{2}{x+1}dx = \lim_{b\to\infty}[2\ln(b/(b+1)) - 2\ln(1/2)] = \ln(4),$$

integral test implies that the given series converges.

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6. (10 pts.) (a) Find the rational number represented by the following repeating decimal.

0.212121212121 ... = 21/99 = 7/33

You may obtain this by either summing the infinite series

$$\sum_{k=1}^{\infty} 21\left(\frac{1}{100}\right)^k = \sum_{j=0}^{\infty} \frac{21}{100}\left(\frac{1}{100}\right)^j$$

or using the "high school" method: $q = 0.\underline{21}$ implies that $100q = 21.\underline{21}$. Thus $99q = 100q - q = 21$. Consequently $q = 21/99$.

(b) Find all values of x for which the given geometric series converges, and then express the closed form sum of the series as a function of x.

Doing the above tasks in reverse order with the hope of being able to pay the piper at the end, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k(x-2)^{k+1}}{10^k} = \sum_{k=0}^{\infty} \frac{(x-2)}{1}\cdot\left(-\left(\frac{x-2}{10}\right)\right)^k = \frac{10(x-2)}{x+8}$$

provided $|(x - 2)/10| < 1$ or $|x - 2| < 10$. As an interval what you have in hand is $I = (-8, 12)$.

7. (10 pts.) Using either comparison test or limit comparison test, determine whether each of the following series converges.

(a) $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^2+1}$ Since

$$\frac{\tan^{-1}(n)}{n^2+1} \le \frac{\pi}{2n^2}$$

for $n \ge 1$, and $\sum_{n=1}^{\infty} \frac{\pi}{2n^2}$ is a positive multiple of a convergent p-series and thus convergent, comparison test implies that the series in (a) converges.

(b) $\sum_{n=1}^{\infty} \frac{\ln(n+2)}{n^{1/2}}$

Since $\ln(n + 2) \ge \ln(e) = 1$ for $n \ge 1$, it follows that

$$\frac{1}{n^{1/2}} \le \frac{\ln(n+2)}{n^{1/2}}$$

for $n \ge 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series, comparison test implies that the series of (b) diverges.

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8. (5 pts.) Using only divergence test, show that the series

$$\sum_{n=1}^{\infty} n \tan\left(\frac{1}{2\pi n}\right)$$

diverges. // Since

$$\lim_{n\to\infty} n \tan\left(\frac{1}{2\pi n}\right) = \lim_{n\to\infty} \left(\frac{1}{2\pi}\right)\left[\frac{\tan\left(\frac{1}{2\pi n}\right)}{\left(\frac{1}{2\pi n}\right)}\right] = \frac{1}{2\pi} \ne 0,$$

divergence test implies that the series immediately above diverges.

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9. (5 pts) With proof, determine whether the given series is conditionally convergent, absolutely convergent, or divergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}$$

Since the series of absolute values is given by

$$\sum_{k=1}^{\infty} \frac{1}{k^4},$$

a convergent p-series, the original series is absolutely convergent.

10. (15 pts.) (a) Using known power series, obtain a power series representation for the function $f(x) = \sin(x^2)$. Write your answer using sigma notation.

$$f(x) = \sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}.$$

(b) Beginning with the power series function

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k x^k$$

defined for x satisfying $|x| < 10$, differentiate termwise to find the series representation for $f'(x)$. Write your answer using sigma notation.

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}\left[\left(\frac{1}{10}\right)^k x^k\right] = \sum_{k=1}^{\infty} k\left(\frac{1}{10}\right)^k x^{k-1}.$$

also for x satisfying $|x| < 10$. Why "k=1"?? Remember our gentle person's agreement with respect to constant terms and sigma notation???

(c) Find a power series representation for the function f(x) below by doing termwise integration. Write your answer using sigma notation.

$$f(x) = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k}\,dt$$
$$= \sum_{k=0}^{\infty} (-1)^k \int_0^x t^{2k}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

for $-1 < x < 1$ by using our friendly geometric series imp.

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11. (5 pts.) Since the third and fourth Maclaurin polynomials for $\sin(x)$ are the same, it follows from Taylor's Theorem that if x is a real number different from 0, we may write

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{\cos(c)}{120}x^5,$$

where c is some number between x and 0. Using this equation, obtain an open interval that is centered at 0 where $\sin(x)$ may be approximated to 1 decimal place accuracy using the polynomial

$$P_4(x) = x - (1/6)x^3.$$

First, from the equation above, it follows that for each real number x different from zero that

$$|\sin(x) - P_4(x)| = |\sin(x) - (x - (1/6)x^3)| = \left|\frac{\cos(c)}{120}x^5\right| \le \frac{|x|^5}{120}.$$

There is no error when x = 0. Thus, to obtain the desired accuracy, then, it suffices to have $|x|^5/120 < 1/20$. Now $|x|^5/120 < 1/20$ is equivalent to $|x| < 6^{1/5}$.
An interval that does the job is $I = (-(6)^{1/5}, (6)^{1/5})$.
Note: Even without a calculator it is very easy to see that the interval above contains the interval J = [-1, 1]. How??

Silly 10 Point Bonus: Obtain the closed form sum of the power series function in Problem 4 on Page 2 of 5. Say where your work is, for it won't fit here.