READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds. Eschew obfuscation.*

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1. (5 pts.) Express the following sum using sigma notation, but do not attempt to find its numerical value.

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} = \sum_{k=2}^{7} \frac{(-1)^k}{k}$$

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2. (5 pts.) Obtain the upper limit of summation and rewrite the summand in order to make the following equation true. **Do not attempt to evaluate the sum.**

$$\sum_{k=4}^{45} k^2 = \sum_{j=1}^{42} (j+3)^2$$

since $j = k - 3$ is equivalent to $k = j + 3$.

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3. (5 pts.) Express the following limit as a definite integral. Do not attempt to evaluate the integral.

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 4x_k^*(1 - 3(x_k^*)^2)\Delta x_k \; ; \quad a = -4,\ b = 1.$$

$$L = \int_{-4}^{1} 4x(1-3x^2)\, dx$$

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4. (10 pts.) If the function f is continuous on [a,b], then the *net signed area* A between $y = f(x)$ and the interval [a,b] is defined by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Reveal all the details in obtaining the numerical value of the net signed area of $f(x) = x^2$ over the interval [0,1] using the definition above with

$$x_k^*$$

the right end point of each subinterval in the regular partition.

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} \\ &= \lim_{n\to\infty} \left(\frac{1}{n^3}\right) \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \left(\frac{n(n+1)(2n+1)}{6n^3}\right) \\ &= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{3} \end{aligned}$$

since $\Delta x = 1/n$ and $x_k = k/n$ for $k = 0, 1, \ldots, n$ are the end points of the intervals of the general regular partition.

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5. (5 pts.) Using a complete sentence and appropriate notation, state precisely the First Part of the Fundamental Theorem of Calculus. [This is sometimes called the *Evaluation Theorem*.]

If f(x) is continuous on [a,b] and g(x) is any antiderivative of f on [a,b], then

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

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6. (5 pts.) Evaluate the following using the 1st Part of the Fundamental Theorem of Calculus.

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

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7. (5 pts.) Using complete sentences and appropriate notation, state the Second Part of the Fundamental Theorem of Calculus.

// Let f(x) be a function that is continuous on an interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\,dt,$$

for each x in I, then g′(x) = f(x) for each x in I.

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8. (10 pts.) A particle moves with a velocity of $v(t) = t^2 - 1$ along an s-axis. Find the displacement and total distance traveled over the time interval [0,2].

$$\text{Displacement} = \int_0^2 v(t)\,dt = \int_0^2 t^2-1\,dt = \ldots = \frac{2}{3}$$

$$\text{Total\_Distance} = \int_0^2 |v(t)|\,dt = \int_0^2 |t^2-1|\,dt = \ldots = 2$$

[See Example 8, Section 6.6 of Anton’s 8th.]

Silly 10 Point Bonus: Show how to use the Mean-Value Theorem of Differential Calculus to prove that

$$(*) \qquad \frac{1}{x+1} < \int_x^{x+1} \frac{1}{t}\,dt < \frac{1}{x}$$

for each $x > 0$. [Say where your work is, for it won’t fit her.]

//Fix $x > 0$. Then f(t) = ln(t) is continuous on the closed interval [x,x+1] and differentiable on the open interval (x,x+1). Applying the Mean-Value Theorem of Differential Calculus to f on [x,x+1], there is a number c in (x,x+1) with

$$\frac{1}{c} = f'(c) = \frac{\ln(x+1)-\ln(x)}{(x+1)-x} = \ln(x+1)-\ln(x)\ .$$

We see that (*) follows since $0 < x < c < x+1$ implies that

$$\frac{1}{x+1} < \frac{1}{c} < \frac{1}{x}$$

and the definition of the natural log implies

$$\int_x^{x+1} \frac{1}{t}\,dt = \ln(x+1)-\ln(x)\ .$$

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9. (5 pts.) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable **t**, so the differential denoting the variable of integration is **dt**.

$$\frac{dy}{dx} = \sec^2(x)e^{\tan(x)} \quad \text{with } y(\pi/4) = 5$$

$$y(x) = \int_{\pi/4}^{x} \sec^2(t)e^{\tan(t)}dt + 5$$

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10. (5 pts.) If the given integral is expressed as an equivalent integral in terms of the variable u using the substitution $u = x^2 - 1$ so that the equation below is true, what are the numerical values of the new limits of integration α and β, and what is the new integrand, f(u) ??? Obtain these but do not attempt to evaluate the du integral.

$$\int_0^2 |x^2-1|2x\,dx = \int_\alpha^\beta f(u)\,du$$

α = -1 β = 3 f(u) = $|u|$

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11. (5 pts.) (a) (3 pts.) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: Complete the sentence,"ln(x) = ...")

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

for $x > 0$. The domain of the natural log function is $(0,\infty)$, and its range is the whole real line, $(-\infty,\infty)$.

(b) (2 pts.) Given ln(a) = -20 and ln(c) = 5, evaluate the following integral.

$$\int_{c^5}^{a} \frac{1}{\lambda}\,d\lambda = \ln(a) - 5\ln(c) = (-20) - (25) = -45$$

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12. (10 pts.) Find each of the following derivatives.

(a) $$\frac{d}{dx}\left[\int_1^x \frac{\sin(t)}{t}\,dt\right] = \frac{\sin(x)}{x}$$

(b) $$\frac{d}{dx}\left[\int_1^{\tan(x)} \tan^{-1}(t)\,dt + x^2\right] = \tan^{-1}(\tan(x))\sec^2(x) + 2x$$

$$= x\sec^2(x) + 2x$$

13. (5 pts.) Evaluate the following limit:

$$L = \lim_{x\to\infty}\left(1 + \frac{1}{3x}\right)^{2x} = \lim_{u\to\infty}\left(1 + \frac{1}{u}\right)^{(2u/3)} = \ldots = e^{\frac{2}{3}}$$

using the substitution $u = 3x$.

14. (10 pts.) Evaluate the following two sums in closed form.

(*a*) $\displaystyle\sum_{k=2}^{20}\left(\frac{1}{k^2} - \frac{1}{(k-1)^2}\right) = \frac{1}{(20)^2} - \frac{1}{1} = -\frac{399}{400}$

(*b*) $\displaystyle\sum_{k=1}^{20} 3^k = \sum_{j=0}^{19} 3^{j+1} = \ldots = \frac{3}{2}[3^{20} - 1]$

15. (10 pts.) Let the function g be defined by the equation

$$g(x) = \int_2^x (3t^2+1)^{1/2}\, dt$$

for $x \in (-\infty, \infty)$. Then since

$$g'(x) = (3x^2+1)^{1/2} \quad\text{and}\quad g''(x) = \frac{3x}{(3x^2+1)^{1/2}},$$

(a) $g(2) = 0$

(b) $g'(2) = \sqrt{13}$

(c) $g''(2) = \dfrac{6}{\sqrt{13}}$

(d) Determine the open intervals where g is increasing or decreasing. Be specific.

By considering the sign of g′ above, it is evident that g is increasing on all of $\mathbb{R} = (-\infty,\infty)$.//

(d) Determine the open intervals where g is concave up or concave down. Be specific.

By considering the sign of g″ above, we can see that g is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.//

Silly 10 Point Bonus: Show how to use the Mean-Value Theorem of Differential Calculus to prove that

$$\frac{1}{x+1} < \int_x^{x+1} \frac{1}{t}\, dt < \frac{1}{x}$$

for each $x > 0$. [Say where your work is, for it won’t fit her.]
[**Proof can be found on the bottom of Page 2 of 4.**]