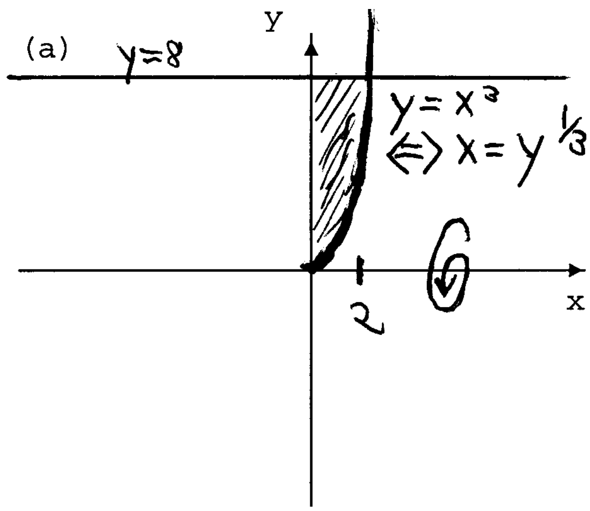
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (25 pts.) (a) Sketch the region R in the first quadrant enclosed by the curves $y = x^3$, $x = 0$, and $y = 8$.



(b) Using the method of disks or washers, write a single definite integral dx whose numerical value is the volume of the solid obtained when the region R above is revolved around the x-axis. Do not evaluate the integral.

$$\text{Volume} = \int_0^2 \pi(8)^2 - \pi(x^3)^2 \, dx$$

(c) Using the method of cylindrical shells, write down a definite integral dy to compute the same volume as in part (b). Do not evaluate the integral.

$$\text{Volume} = \int_0^8 2\pi y \cdot y^{1/3} \, dy$$

(d) Write down, but do not attempt to evaluate the definite integral whose numerical value gives the area of the region R above if one integrates with respect to y.

$$\text{Area} = \int_0^8 y^{1/3} \, dy$$

(e) Write down, but do not attempt to evaluate, the definite integral that gives the arc-length of the curve $y = (2/3)x^{3/2} + 5$ from $x = 0$ to $x = 2$.

$$\text{Length} = \int_0^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx = \int_0^2 (1 + x)^{1/2} \, dx$$

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2. (10 pts.) Consider the definite integral below. (a) Write down the sum, S_4, used to approximate the value of the integral below if Simpson's Rule is used with n = 4. **Do not attempt to evaluate the sum. Be very careful.** (b) Write down the sum, T_4, used to approximate the value of the integral below if Trapezoid Rule is used with n = 4. **Do not attempt to evaluate the sum. Be very careful.**

$$\int_{10}^{12} x^{1/8}\,dx$$

$$\text{Plainly,} \quad \Delta x = \frac{1}{2},\ and\ x_k = 10 + \frac{k}{2} = \frac{20+k}{2},\ for\ k = 0,1,2,3,4$$

are the points of the regular partition we need.

(a)

$$\begin{aligned} S_4 &= \frac{1}{3}\Delta x(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\ &= \frac{1}{6}\left((\tfrac{20}{2})^{1/8} + 4(\tfrac{21}{2})^{1/8} + 2(\tfrac{22}{2})^{1/8} + 4(\tfrac{23}{2})^{1/8} + (\tfrac{24}{2})^{1/8} \right) \end{aligned}$$

(b)

$$\begin{aligned} T_4 &= \frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{1}{4}\left((\tfrac{20}{2})^{1/8} + 2(\tfrac{21}{2})^{1/8} + 2(\tfrac{22}{2})^{1/8} + 2(\tfrac{23}{2})^{1/8} + (\tfrac{24}{2})^{1/8} \right) \end{aligned}$$

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3. (15 pts.) (a) (10 pts.) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc. Be very careful here.

$$\frac{2x^2+4}{(x+1)^2(4x^2+9)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{4x^2+9} + \frac{Ex+F}{(4x^2+9)^2} + \frac{Gx+H}{(4x^2+9)^3}$$

(b) (5 pts.) If one were to integrate the rational function in part (a), one probably would encounter the integral below. Reveal, in detail, how to evaluate this integral.

$$\int \frac{1}{4x^2+9}\,dx = \frac{1}{9}\int \frac{1}{(2x/3)^2+1}\,dx = \frac{1}{6}\int \frac{1}{1+u^2}\,du = \frac{1}{6}\tan^{-1}(u) + C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + C$$

by using the substitution $u = 2x/3$. You may also use the trigonometric substitution $\tan(\theta) = 2x/3$, but doing so will cost additional work.

Reconciliation Hint: To see that

$$\ln\left|\frac{\tan(\frac{x}{2})+1}{\tan(\frac{x}{2})-1}\right| = \ln|\sec(x)+\tan(x)|,$$

convert the tangents to sines and cosines, clear the common denonminator in the complex fraction that results, and then multiply the numerator and denominator within the absolute value bars by $\sin(x/2) + \cos(x/2)$. Finish things off by using Pythagoras suitably, and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, and $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ with $\theta - x/2$, yadda, yadda, yadda.

4. (50 pts.) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[**5 pts./part**]

(a) $\int_0^{(\pi/3)^{1/2}} (4x)\cos(x^2)\, dx = \int_0^{\pi/3} 2\cos(u)\, du = 2\sin(u)\big|_0^{\pi/3} = 2\sin(\pi/3) = \sqrt{3}$

using the u-substitution $u = x^2$.

(b) $\int \sin^{-1}(x)\, dx = x\sin^{-1}(x) - \int \frac{x}{(1-x^2)^{1/2}}\, dx = x\sin^{-1}(x) + (1-x^2)^{1/2} + C$

by integrating by parts using $u = \sin^{-1}(x)$ and $dv = 1\cdot dx$, and then performing an obvious u-substitution.

(c) $\int 4x\cos(x)\, dx = 4x\sin(x) - \int 4\sin(x)\, dx = 4x\sin(x) + 4\cos(x) + C$

by integrating by parts using $u = 4x$ and $dv = \cos(x)dx$.

(d) $\int \frac{\cos^2(t)}{\sin(t)}\, dt = \int csc(t) - \sin(t)\, dt = -\ln|csc(t) + \cot(t)| + \cos(t) + C$

after disinterring Pythagoras. [If you cannot fill in the *missing step*, go directly to *Appendix A*. Do not pass go, etc.]

(e) $\int (4x+4)e^{2x}\, dx = (4x+4)\frac{e^{2x}}{2} - \int 2e^{2x}\, dx = (2x+2)e^{2x} - e^{2x} + C$

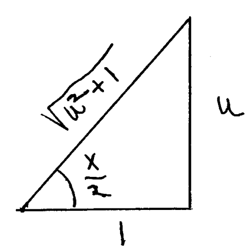
by integrating by parts using $u = 4x + 4$ and $dv = e^{2x}\, dx$. You could, of course, be more clever and steal the needed factor of 2 from $4x + 4$ to make things cleaner.

Silly 10 Point Bonus: Show how to evaluate the integral

$$\int \sec(x)\, dx$$

using the substitution $u = \tan(x/2)$, and then reconcile the result you get with the usual integral for secant.

From the the right triangle to the right, you may read off that $\sin(x/2) = u/(u^2 + 1)^{1/2}$ and that $\cos(x/2) = 1/(u^2 + 1)^{1/2}$. Applying appropriate double angle trig magic then yields the following two formulae: $\sin(x) = 2u/(u^2 + 1)$ and $\cos(x) = (1 - u^2)/(u^2 + 1)$ And finally, since $u = \tan(x/2)$ implies $x = 2\tan^{-1}(u)$, $dx = (2/(u^2 + 1))du$. Now put all this to work to evaluate the integral.



$$\int \sec(x)\, dx = \int \frac{1}{\cos(x)}\, dx = \int \frac{1+u^2}{1-u^2}\cdot\frac{2}{1+u^2}\, du = \int \frac{-2}{(u+1)(u-1)}\, du$$
$$= \int \frac{1}{u+1} - \frac{1}{u-1}\, du = \ln\left|\frac{u+1}{u-1}\right| + C$$
$$= \ln\left|\frac{\tan(\frac{x}{2})+1}{\tan(\frac{x}{2})-1}\right| + C$$

after a partial-fraction decomposition. **Reconciliation Hint**: Page 2.

4. (Continued) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.

(f)

$$\int \frac{1}{(x^2-1)^{1/2}}\,dx = \int \frac{\sec(\theta)\tan(\theta)}{|\tan(\theta)|}\,d\theta = \int \sec(\theta)\,d\theta$$

$$= \ln|\sec(\theta)+\tan(\theta)| + C = \ln|x+(x^2-1)^{1/2}| + C$$

using the trigonometric substitution $x = \sec(\theta)$ and $x > 1$. When $x < -1$, the integral that results is actually $-\ln|x - (x^2 - 1)^{1/2}| + C$, which is easily seen to be a disguised form of $\ln|x + (x^2 - 1)^{1/2}| + C$ after a bit of obvious algebraic shenanegans. [It turns out that $\tan(\theta)$ is negative when $\pi/2 < \theta < \pi$.]

(g) $$\int_0^1 \frac{2x^3+4x}{x^2+1}\,dx = \int_0^1 2x + \frac{2x}{x^2+1}\,dx = \left(x^2+\ln(x^2+1)\right)\Big|_0^1 = \ldots = 1+\ln(2)$$

after doing an easy long division.

(h)

$$\int (1-t^2)^{1/2}\,dt = \int \cos^2(\theta)\,d\theta = \int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C = \frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2} + C$$

$$= \frac{1}{2}\left(\sin^{-1}(t) + t(1-t^2)^{1/2}\right) + C$$

using the obvious trigonometric substitution $t = \sin(\theta)$. Of course you could also try integration by parts, but that route is a little thornier.

(i) $$\int \frac{2x+1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C$$

after performing an easy partial-fraction decomposition.

(j)

$$\int \sin(x)e^x\,dx = \sin(x)e^x - \int \cos(x)e^x\,dx$$

$$= \sin(x)e^x - \left(\cos(x)e^x - \int (-1)\sin(x)e^x\,dx\right)$$

by integrating by parts twice in succession, all the while continually picking on our beloved exponential function as the recognized derivative. Solving this little linear equation allows us to write

$$\int \sin(x)e^x\,dx = \left(\frac{\sin(x)-\cos(x)}{2}\right)e^x + C.$$

Silly 10 Point Bonus: Show how to evaluate the integral

$$\int \sec(x)\,dx$$

using the substitution $u = \tan(x/2)$, and then reconcile the result you get with the usual integral for secant. Say where your work is, for there isn't room here. **Look on pages 3 and 2, in that order. o.o.**