READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page. Eschew obfuscation.*

1. (10 pts.) Obtain the second Taylor polynomial $p_2(x)$ of the function

$$f(x) = x^{1/2}$$

at $x_0 = 4$.

Plainly,

$$f'(x) = (1/2)x^{-1/2} \text{ and } f''(x) = -(1/4)x^{-3/2}$$

for $x > 0$. Thus,

$$f^{(0)}(4) = 2,\ f^{(1)}(4) = \frac{1}{4}, \text{ and } f^{(2)}(4) = -(1/4)(4)^{-3/2} = -\frac{1}{32}.$$

Consequently, the second Taylor polynomial at $x_0 = 4$ is

$$p_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(4)}{k!}(x-4)^k = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2.$$

2. (10 pts.) Suppose that

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k\,10^k}(x-1)^k$$

Find the radius of convergence and the interval of convergence of the power series function f.

// To use ratio test for absolute convergence, we compute

$$\rho(x) = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)\frac{1}{10}|x-1| = \frac{1}{10}|x-1|.$$

Plainly,

$$\rho(x) < 1 \iff \frac{1}{10}|x-1| < 1 \iff |x-1| < 10.$$

Thus, the radius of convergence is R = 10. By unwrapping the rightmost inequality above, we can obtain the interior of the interval of convergence, namely, the interval (-9,11). Substitution of $x = -9$ into f yields

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

which diverges. The substitution of $x = 11$ into f yields

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

which is a convergent alternating series. Consequently, the interval of convergence is I = (-9,11].

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3. (10 pts.) Each of the following power series functions is the Maclaurin series of some well-known function. In each case, (i) identify the function, and (ii) provide the interval in which the series actually converges to the function.

(a) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin(x)$ *for* $x \in \Re = (-\infty,\infty)$.

(b) $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ *for* $x \in \Re = (-\infty,\infty)$.

(c) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = \tan^{-1}(x)$ *for* $x \in [-1,1]$.

(d) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \cos(x)$ *for* $x \in \Re = (-\infty,\infty)$.

(e) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = \ln(1+x)$ *for* $x \in (-1,1]$.

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4. (5 pts.) Express .272727... (repeating) as the ratio of two positive integers. [**The ratio does not have to be in lowest terms.**]

.272727... = 27/99 = 3/11

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5. (5 pts.) Prove the infinite series below is conditionally convergent. [Helpful Hint ?? : $\ln(k) < k$ when $k \geq 1$.]

(*) $$\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{\ln(k)}$$

First, the series of absolute values is given by

(**) $$\sum_{k=2}^{\infty} \left| \frac{(-1)^{k+1}}{\ln(k)} \right| = \sum_{k=1}^{\infty} \frac{1}{\ln(k)}.$$

By using the helpful hint, it follows that

$$\frac{1}{k} < \frac{1}{\ln(k)}$$

for $k \geq 2$. [*Watch that division by zero problem, Folks!*] Since the harmonic series diverges, by using comparison test with the appropriate tail of the harmonic, it follows that (**) above diverges. Hence (*) above is not absolutely convergent. Obviously (*) is an alternating series. Observe that since your friendly natural logarithm is an increasing function and positive for $k \geq 2$,

$$\frac{1}{\ln(k)} > \frac{1}{\ln(k+1)} \text{ for } k \geq 2, \text{ and } \lim_{k \to \infty} \frac{1}{\ln(k)} = 0$$

since $\ln(k) \to \infty$ as $k \to \infty$. Thus, alternating series test implies that (*) converges. Since (*) converges, but not absolutely, (*) is conditionally convergent.

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6. (5 pts.) Using root test, determine whether the following series converges.

$$\sum_{k=1}^{\infty}\left(\frac{2}{1+\frac{1}{k^2}}\right)^k$$

Since $\rho = \lim_{k\to\infty}\left[\left(\frac{2}{1+(1/k^2)}\right)^k\right]^{1/k} = \lim_{k\to\infty}\frac{2}{1+(1/k^2)} = 2 > 1$,
root test implies that the series above diverges.//

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7. (5 pts.) Using comparison test or limit comparison test, determine whether the following series converges.

$$\sum_{k=1}^{\infty}\frac{10k+1}{2+3k^3}$$

Since $\frac{10k+1}{2+3k^3} \le \frac{11}{k^2}$ for $k \ge 1$, and $\sum_{k=1}^{\infty}\frac{11}{k^2}$ is a nonzero multiple of a convergent p-series, and so convergent, comparison test implies that the series above converges.//

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8. (5 pts.) Does divergence test tell you anything about the following series? Explain briefly. Note: You do not actually have to determine whether the series converges.

$$\sum_{k=1}^{\infty}\frac{10\ln(k)}{2+3k}$$

Since $\lim_{k\to\infty}\frac{10\ln(k)}{2+3k} = 0$, divergence test yields no information about the convergence of the series above. That the sequence of terms have limit zero is a necessary, but not a sufficient condition for the convergence of the series.//

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9. (5 pts.) Using integral test, determine whether the following series converges. [Hint??: Begin by defining f(x) appropriately.]

$$\sum_{k=2}^{\infty}\frac{4}{k\ln(k)}$$

First, set $f(x) = \frac{4}{x\ln(x)}$ for $x \in [2,\infty)$. Plainly, f is positive and continuous on the interval $[2,\infty)$. Since

$$f'(x) = \frac{-4(\ln(x)+1)}{(x\ln(x))^2} < 0 \text{ for } x \in (2,\infty),$$

f is decreasing on $[2,\infty)$. We may use integral test now. Since

$$\int_2^{\infty}\frac{4}{x\ln(x)}dx = \lim_{b\to\infty}\int_2^{b}\frac{4}{x\ln(x)}dx = \lim_{b\to\infty}[\,4\ln(\ln(b)) - 4\ln(\ln(2))\,] = \infty,$$

integral test implies that this series diverges.//

10. (20 pts.) Obtain the exact numerical value of each of the following if possible. If a limit doesn't exist or an improper integral or an infinite series fails to converge, say so.

(a) $$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}(-e^{-x})\Big|_1^b = \lim_{b\to\infty}(e^{-1} - e^{-b}) = \frac{1}{e}.$$

(b) $$\sum_{k=1}^{\infty} e^{-k} = \sum_{j=0}^{\infty} e^{-(j+1)} = \sum_{j=0}^{\infty}\frac{1}{e}\left(\frac{1}{e}\right)^j = \frac{\left(\frac{1}{e}\right)}{1-\left(\frac{1}{e}\right)} = \frac{1}{e-1}$$

(c) $$\int_1^2 \frac{1}{(2-x)^{1/2}}\,dx = \lim_{b\to 2^-}\int_1^b \frac{1}{(2-x)^{1/2}}\,dx = \lim_{b\to 2^-}\int_1^{2-b}\frac{-1}{u^{1/2}}\,du$$

$$= \lim_{b\to 2^-}\int_{2-b}^1 u^{-1/2}\,du = \lim_{b\to 2^-}[\,2(1)^{1/2} - 2(2-b)^{1/2}\,] = 2.$$

by using the substitution $u = 2 - x$.

(d) $$\lim_{n\to\infty}\cos\left(\frac{\ln(n)}{n}\right) = \cos(0) = 1$$

since cosine is continuous and $\lim_{n\to\infty}\frac{\ln(n)}{n} = 0$.

This last limit may be seen in a variety of ways. You could use l'Hopital's Rule with the real version of the limit. You could also be cute and squeeze by a making a clever use of the inequality $\ln(x) < x$ for $x \geq 1$.

Silly 10 Point Bonus: Show how to find an interval that is symmetric about the origin where $\sin(x)$ can be approximated by $p(x) = x - x^3/6$ with two decimal place accuracy. [Indicate where your work is, for it won't fit here.]

It turns out that $p(x)$ above is both the third and fourth Maclaurin polynomials for $\sin(x)$. Consequently, since the fifth derivative of sin is bounded in magnitude by 1, it follows from the Remainder Estimation Theorem that if x is any real number, we may write

$$|\sin(x) - p(x)| = |\sin(x) - (x - (1/6)x^3)| \leq \frac{|x|^5}{120}.$$

Thus, to obtain the desired accuracy, then, it suffices to have

$$\frac{|x|^5}{120} < \frac{1}{2}10^{-2}.$$

Now

$$\frac{|x|^5}{120} < \frac{1}{2}10^{-2} \Leftrightarrow |x| < (.6)^{1/5}.$$

An interval that does the job is $I = (-(.6)^{1/5}, (.6)^{1/5})$.

Note: Even without a calculator it is very easy to see that the interval above contains the interval $J = (-0.6, 0.6)$. How??

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11. (6 pts.) Suppose

$$f(x) = \sum_{k=1}^{\infty} \frac{\pi(x-5)^k}{k\,20^k}$$

for every $x \in (-15,25)$. By differentiating f term-by-term, obtain a power series function that is the same as f′(x). **Use sigma notation as in class.**

$$f'(x) = \sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{\pi(x-5)^k}{k20^k}\right] = \sum_{k=1}^{\infty} \frac{\pi k(x-5)^{k-1}}{k20^k} = \sum_{k=1}^{\infty} \frac{\pi(x-5)^{k-1}}{20^k}.$$

This is an old geometric beast that you can actually sum. Furthermore, you should be able to figure out a useful alias for f.

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12. (6 pts.) **Using sigma notation as in class and an appropriate Maclaurin series**, by doing term-by-term integration, obtain an infinite series that is equal to the numerical value of the following definite integral.

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}\,dx = \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k x^{2k}}{k!}\,dx$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 x^{2k}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}.$$

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13. (8 pts.) (a) By substitution into an appropriate Maclaurin series, obtain the Maclaurin series for the function

$$f(x) = \frac{18}{9+x^2}$$

[**Hint**: *You must do a little algebra before substituting. Expect to meet an old friend??*]
(b) What is the domain of the function f?
(c) What is the interval of convergence for the Maclaurin series of f??

(a) $$f(x) = \frac{18}{9+x^2} = \frac{2}{1-\left(-\frac{x^2}{9}\right)} = 2\sum_{k=0}^{\infty}\left(\frac{(-1)x^2}{9}\right)^k = \sum_{k=0}^{\infty}\frac{2(-1)^k x^{2k}}{9^k}$$

provided that

$$\left|\frac{(-1)x^2}{9}\right| < 1.$$

(b) Plainly, the domain of f is $\mathbb{R} = (-\infty,\infty)$.
(c) The Maclaurin series is a geometric varmint. Consequently, the interval of convergence may be read off of the inequality above with no additional real work: I = (-3,3)

Silly 10 Point Bonus: Show how to find an interval that is symmetric about the origin where sin(x) can be approximated by p(x) = $x - x^3/6$ with two decimal place accuracy. [Indicate where your work is, for it won't fit here.] This may be found on Page 4 of 5.