

READ ME FIRST: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals", " \Rightarrow " denotes "implies", and " \Leftrightarrow " denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds. Eschew obfuscation.

1. (5 pts.) Express the following sum using sigma notation, but do not attempt to find its numerical value.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} = \sum_{k=1}^7 \frac{(-1)^{k+1}}{k}$$

Observe that this equation may be completed in infinitely many correct ways.

2. (5 pts.) Obtain the upper limit of summation and rewrite the summand in order to make the following equation true. **Do not attempt to evaluate the sum.**

$$\sum_{k=1}^{45} k^4 = \sum_{j=10}^{54} (j-9)^4$$

since $j = k + 9$ is equivalent to $k = j - 9$.

3. (5 pts.) Using a complete sentence and appropriate notation, state precisely the First Part of the Fundamental Theorem of Calculus. [This is sometimes called the *Evaluation Theorem*.]

If $f(x)$ is continuous on $[a,b]$ and $g(x)$ is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x) \, dx = g(b) - g(a).$$

4. (5 pts.) Evaluate the following using the 1st Part of the Fundamental Theorem of Calculus.

$$\int_{-\pi/4}^{\pi/4} \cos(x) \, dx = \sin(x) \Big|_{-\pi/4}^{\pi/4} = \sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}.$$

5. (5 pts.) Express the following limit as a definite integral. Do not attempt to evaluate the integral.

$$L = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n (\sin^2 x_k^*) \Delta x_k ; \quad a = 0, \quad b = \pi/2.$$

$$L = \int_0^{\pi/2} \sin^2(x) \, dx$$

Silly 10 Point Bonus: Obtain the exact numerical value of the limit L in Problem 5 on Page 1 of 4.

$$L = \int_0^{\pi/2} \sin^2(x) \, dx = \int_0^{\pi/2} \frac{1 - \cos(2x)}{2} \, dx = \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right) \Big|_0^{\pi/2} = \dots = \frac{\pi}{4}$$

Trig or treat!! [Integration by parts is less accessible at this time.]

6. (5 pts.) Using complete sentences and appropriate notation, state the Second Part of the Fundamental Theorem of Calculus.

// Let $f(x)$ be a function that is continuous on an interval I , and suppose that a is any point in I . If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t) dt,$$

for each x in I , then $g'(x) = f(x)$ for each x in I .

7. (10 pts.) Find each of the following derivatives.

(a) $\frac{d}{dx} \left[\int_1^x \sin(t^2) dt \right] = \sin(x^2)$

(b) $\frac{d}{dx} \left[\int_0^{e^{-x}} \frac{1}{1+t^4} dt \right] = \frac{1}{1+(e^{-x})^4} \cdot (-1)e^{-x} = \frac{-e^{-x}}{1+e^{-4x}}$

8. (5 pts.) (a) (3 pts.) Give the definition of the function $\ln(x)$ in terms of a definite integral and give its domain and range. Label correctly. (**Hint:** Complete the sentence, " $\ln(x) = \dots$ ".)

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

for $x > 0$. The domain of the natural log function is $(0, \infty)$, and its range is the whole real line, $(-\infty, \infty)$.

(b) (2 pts.) Given $\ln(a) = 2$ and $\ln(b) = -3$, evaluate the following integral.

$$\int_1^{ab^2} \frac{1}{t} dt = \ln(ab^2) = \ln(a) + \ln(b^2) = \ln(a) + 2\ln(b) = 2 + 2(-3) = -4.$$

9. (5 pts.) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable t , so the differential denoting the variable of integration is dt . DO NOT ATTEMPT TO EVALUATE THE DEFINITE INTEGRAL IN t THAT YOU OBTAIN.

$$\frac{dy}{dx} = \frac{2x^2+1}{x}, \quad y(1) = 2.$$

$$y(x) = \int_1^x \frac{2t^2+1}{t} dt + 2$$

10. (5 pts.) Evaluate the definite integral below by expressing it in terms of u correctly and then evaluating the resulting integral using a formula from geometry.

$$\int_{-5/3}^{5/3} \sqrt{25-9x^2} \, dx; \, u = 3x$$

$$\int_{-5/3}^{5/3} \sqrt{25-9x^2} \, dx = \int_{-5}^5 \sqrt{25-u^2} \frac{1}{3} du = \frac{1}{3} \int_{-5}^5 \sqrt{25-u^2} \, du = \frac{25\pi}{6}$$

since $du = 3dx$. The right-most integral provides the area of half a circle with a radius of 5.

11. (10 pts.) Write each of the following two sums in closed form.

$$(a) \sum_{k=1}^n \frac{3k}{n} = \frac{3}{n} \sum_{k=1}^n k = \frac{3}{2}(n+1)$$

$$(b) \sum_{k=0}^n \frac{1}{2^k} = \sum_{k=0}^n \left(\frac{1}{2}\right)^k = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 2 \left[1 - \left(\frac{1}{2}\right)^{n+1}\right].$$

12. (10 pts.) If the function f is continuous on $[a,b]$, then the *net signed area* A between $y = f(x)$ and the interval $[a,b]$ is defined by

$$A = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x.$$

Reveal all the details in computing the numerical value of the net signed area of $f(x) = x^2$ over the interval $[0,2]$ using only the definition above with

$$x_k^*$$

the right end point of each subinterval in the regular partition.

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{2k}{n}\right)^2 \frac{2}{n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{8}{n^3}\right) \sum_{k=1}^n k^2 = \lim_{n \rightarrow \infty} \left(\frac{8n(n+1)(2n+1)}{6n^3}\right) \\ &= \lim_{n \rightarrow \infty} \frac{8}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) = \frac{8}{3} \end{aligned}$$

since $\Delta x = 2/n$ and $x_k = 2k/n$ for $k = 0, 1, \dots, n$ are the end points of the intervals of the general regular partition.

13. (10 pts.) A particle moves with a velocity of $v(t) = \cos(t)$ along an s-axis. Find the displacement and total distance traveled over the time interval $[\pi/2, 2\pi]$.

$$\text{Displacement} = \int_{\pi/2}^{2\pi} v(t) \, dt = \int_{\pi/2}^{2\pi} \cos(t) \, dt = \dots = -1$$

$$\text{Total_Distance} = \int_{\pi/2}^{2\pi} |v(t)| \, dt = \int_{\pi/2}^{2\pi} |\cos(t)| \, dt = \dots = 3$$

14. (5 pts.) Evaluate the following limit:

$$L = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{2x}\right)^x = \lim_{u \rightarrow \infty} \left(1 + \frac{1}{u}\right)^{(u/2)} = \dots = e^{\frac{1}{2}}$$

using the substitution $u = 2x$.

15. (10 pts.) Let the function g be defined by the equation

$$g(x) = \int_0^x \cos^4(t) \, dt + 2$$

for $x \in (-\pi/2, \pi/2)$. Then since

$$g'(x) = \cos^4(x) \quad \text{and} \quad g''(x) = -4 \sin(x) \cos^3(x) ,$$

$$(a) \quad g(0) = \int_0^0 \cos^4(t) \, dt + 2 = 0 + 2 = 2$$

$$(b) \quad g'(0) = 1$$

$$(c) \quad g''(0) = 0$$

(d) Determine the open intervals where g is increasing or decreasing. Be specific.

By considering the sign of g' above, it is evident that g is increasing on all its domain, $(-\pi/2, \pi/2)$. //

(e) Determine the open intervals where g is concave up or concave down. Be specific.

By considering the sign of g'' above, we can see that g is concave up on $(-\pi/2, 0)$ and concave down on $(0, \pi/2)$. Hint: cosine is positive on the interval $(-\pi/2, \pi/2)$. $\sin(x)$ is the determining factor. //

Silly 10 Point Bonus: Obtain the exact numerical value of the limit L in Problem 5 on Page 1 of 4. [Say where your work is, for it won't fit here.]

This may be found on the bottom of Page 1 of 4.