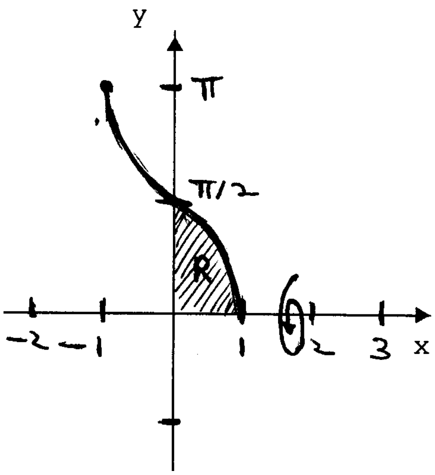
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (25 pts.) The region in the first quadrant enclosed by the curves $y = \cos^{-1}(x)$, $x = 0$, and $y = 0$ is sketched below for your convenience.



(a) Write down, but do not attempt to evaluate the definite integral whose numerical value gives the area of the region R if one integrates with respect to x so the differential in the integral is dx.

$$\text{Area} = \int_0^1 \cos^{-1}(x)\, dx$$

(b) Write down, but do not attempt to evaluate the definite integral whose numerical value gives the area of the region R if one integrates with respect to y so the differential in the integral is dy.

$$\text{Area} = \int_0^{\pi/2} \cos(y)\, dy$$

(c) Using the method of disks or washers, write a single definite integral dx whose numerical value is the volume of the solid obtained when the region R above is revolved around the x-axis. Do not evaluate the integral.

$$\text{Volume} = \int_0^1 \pi(\cos^{-1}(x))^2\, dx$$

(d) Using the method of cylindrical shells, write down a definite integral dy to compute the same volume as in part (c). Do not evaluate the integral.

$$\text{Volume} = \int_0^{\pi/2} 2\pi y \cos(y)\, dy$$

(e) Write down, but do not attempt to evaluate, the definite integral that gives the arc-length of the curve $y = \cos^{-1}(x)$ from $x = 0$ to $x = 1/2$.

$$\text{Length} = \int_0^{1/2} \sqrt{1 + (dy/dx)^2}\, dx = \int_0^{1/2} \sqrt{1 + \left(-\frac{1}{\sqrt{1-x^2}}\right)^2}\, dx$$

$$= \int_0^{1/2} \sqrt{\frac{2-x^2}{1-x^2}}\, dx$$

or

$$\text{Length} = \int_{\cos^{-1}(1/2)}^{\cos^{-1}(0)} \sqrt{1 + (dx/dy)^2}\, dy = \int_{\pi/3}^{\pi/2} \sqrt{1 + (-\sin(y))^2}\, dy$$

$$= \int_{\pi/3}^{\pi/2} \sqrt{1 + \sin^2(y)}\, dy$$

2. (15 pts.) (a) (10 pts.) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc. Be very careful here.

$$\frac{4x^2+5}{(x+1)^3(9x^2+4)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{9x^2+4} + \frac{Fx+G}{(9x^2+4)^2}$$

(b) (5 pts.) If one were to integrate the rational function in part (a), one probably would encounter the integral below. Reveal, in detail, how to evaluate this integral.

$$\int \frac{1}{9x^2+4}\,dx = \frac{1}{4}\int \frac{dx}{1+\left(\frac{3x}{2}\right)^2} = \frac{1}{6}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{6}\tan^{-1}(u) + C = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C$$

using the substitution $u = 3x/2$. You could also use the trig substitution $\tan(\theta) = 3x/2$ to beat up on this varmint.

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3. (60 pts.) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[**6 pts./part**]

(a) $\int 2x\cos(x)\,dx = 2x\sin(x) - \int 2\sin(x)\,dx = 2x\sin(x) + 2\cos(x) + C$

by integrating by parts using $u = 2x$ and $dv = \cos(x)dx$.

(b)

$$\int_0^{(\pi/2)^{1/2}} 2x\sin(x^2)\,dx = \int_0^{\pi/2} \sin(u)\,du = (-\cos(u))\Big|_0^{\pi/2}$$

$$= -\cos(\pi/2) - (-\cos(0)) = 1$$

using the u-substitution $u = x^2$.

Silly 10 Point Bonus: What magical theorem ensures that all real functions f that are continuous on an interval, I, have real honest-to-goodness antiderivatives that are alive and well on the interval, I?? State and prove the magical theorem.//

What is being sought is the second part of the Fundamental Theorem of Calculus. Perhaps most simply it could be stated thus:

Let f be a function that is continuous on a non-degenerate interval I, and let a be a number in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\,dt,$$

for each x in I, then $g'(x) = f(x)$ for each x in I.

You can, of course, find a proof of this in Anton's ET 8th Edition on about page 404 that uses the Mean-Value Theorem for Integrals. Think of it as a squeeze in time saves nine. You don't have to use the Mean-Value Theorem for Integrals if you are willing to shoulder some epsilon antics. See the bottom of page 3.

3. (Continued) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[6 pts./part]

(c) $$\int \tan^{-1}(x)\, dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\, dx = x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + c$$

by integrating by parts using $u = \tan^{-1}(x)$ and $dv = 1 \cdot dx$, and then performing an obvious u-substitution.

(d) $$\int x^2 e^x dx = x^2 e^x - \int 2xe^x\, dx = \ldots = x^2e^x - 2xe^x + 2e^x + C$$

by integrating by parts twice in succession, all the while continually picking on our really beloved exponential function as the recognized derivative.

(e) $$\int \frac{\sin^2(t)}{\cos(t)}\, dt = \int \sec(t) - \cos(t)\, dt$$
$$= \ln|\sec(t) + \tan(t)| - \sin(t) + C$$

after disinterring Pythagoras. [If you cannot fill in the *missing step* by using a suitable trig identity or two, go directly to *Appendix A*. Do not pass go, etc.]

(f) $$\int \sqrt{1 - x^2}\, dt = \int \cos^2(\theta)\, d\theta = \int \frac{1+\cos(2\theta)}{2}\, d\theta$$
$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C = \frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2} + C$$
$$= \frac{1}{2}\left(\sin^{-1}(x) + x\sqrt{1-x^2}\right) + C$$

using the obvious trigonometric substitution $x = \sin(\theta)$. Of course you could also try integration by parts, but that route is a little thornier.

Bonus: [Continued.] Suppose x is an interior point of I and f is continuous at x. Let $\varepsilon > 0$. If f is continuous at x, there is a small number $\delta > 0$ so that if $|t - x| < \delta$, then $|f(t) - f(x)| < \varepsilon/2$. Grab this wee δ. Suppose that we have $0 < |h| < \delta$.

If $0 < h < \delta$, then $[x,x+h] \subset (x-\delta,x+\delta)$. Consequently,

$$\left|\frac{g(x+h) - g(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} f(t) - f(x)\, dt\right|$$
$$\leq \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\, dt$$
$$\leq \frac{1}{h}\int_x^{x+h} \frac{\varepsilon}{2}\, dt < \varepsilon.$$

If $-\delta < h < 0$, then we also have $[x+h,x] \subset (x-\delta,x+\delta)$. Thus,

$$\left|\frac{g(x+h) - g(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} f(t) - f(x)\, dt\right|$$
$$= \left|\frac{1}{h}\int_{x+h}^{x} f(t) - f(x)\, dt\right|$$
$$\leq \frac{1}{|h|}\int_{x+h}^{x} |f(t)-f(x)|\, dt$$
$$\leq \frac{1}{|h|}\int_{x+h}^{x} \frac{\varepsilon}{2}\, dt < \varepsilon.$$

It now follows from the silly ε-δ definition of the desired limit that we are well-cooked.//

3. (Continued) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.

(g) $\int_0^1 \frac{2x^4+4x^3}{x^2+1}\,dx = \int_0^1 2x^2 + 4x - 2 - \frac{4x-2}{x^2+1}\,dx = \ldots = \frac{2}{3} - 2\ln(2) + \frac{\pi}{2}$

after doing an easy long division.

(h) $\int \sin(4t)\cos(2t)\,dt = \int 2\sin(2t)\cos^2(2t)\,dt = -\frac{\cos^3(2t)}{3} + c$

using the trig identity sin(4t) = 2sin(2t)cos(2t) and an obvious u-substitution.

$$\int \sin(4t)\cos(2t)\,dt = \frac{1}{2}\int \sin(6t) + \sin(2t)\,dt$$
$$= -\frac{1}{12}\cos(6t) - \frac{1}{4}\cos(2t) + c$$

where we have used that identity

$$\sin(4t)\cos(2t) = (1/2)[\sin(4t+2t) + \sin(4t-2t)]$$

which may be obtained in real time quickly from identities you should have stored in your bio-computer.

Note: This antiderivative can be obtained in at least three different ways. Two involve the easy use of trigonometric identities, and the third is a thorny integration by parts. Above, you will have seen the easy trig or treat routes.

(i) $\int \frac{1}{x^2-1}\,dx = \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$

after performing an easy partial-fraction decomposition.

(j)

$$\int \cos(x)e^x\,dx = \cos(x)e^x - \int (-\sin(x))e^x\,dx$$
$$= \cos(x)e^x + \left(\sin(x)e^x - \int \cos(x)e^x dx\right)$$

by integrating by parts twice in succession, all the while continually picking on our beloved exponential function as the recognized derivative. Solving this little linear equation allows us to write

$$\int \sin(x)e^x\,dx = \left(\frac{\sin(x)+\cos(x)}{2}\right)e^x + C.$$

Silly 10 Point Bonus: What magical theorem ensures that all real functions *f* that are continuous on an interval, I, have real honest-to-goodness antiderivatives that are alive and well on the interval, I?? State and prove the magical theorem.//Say where your work is, for there isn't room here. *Look on pages 2 and 3*.