READ ME FIRST: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page. Eschew obfuscation.

1. (6 pts.) (a) Using complete sentences and appropriate notation, give the precise ε - N definition of

(*) $$\lim_{n\to\infty} a_n = L.$$

We write (*) above if L is a number such that, for each $\varepsilon > 0$, there is a positive integer N, dependent on ε, such that for every positive integer n, if $n \geq N$, then $|a_n - L| < \varepsilon$.

(b) Give the precise mathematical definition of the sum of an infinite series,

(**) $$\sum_{k=1}^{\infty} a_k$$

A number s is the sum of the series (**) above if

$$s = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

If the limit fails to exist, the series is said to diverge.

2. (2 pts.) Suppose that

$$\sum_{n=1}^{\infty} a_n = -\frac{2}{15} \qquad \text{and} \qquad \sum_{n=1}^{\infty} b_n = 6.$$

Then we have

$$\sum_{n=1}^{\infty}\left(60a_n + \frac{5}{3}b_n\right) = \ldots = 60\left(-\frac{2}{15}\right) + \frac{5}{3}(6) = 2.$$

3. (12 pts.) (a) [Complete the following.] A p-series is a series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

This series converges if ____ $p > 1$ ____ and this series diverges if

____ $p \leq 1$ ____.

(b) [Complete the following.] A geometric series is a series of the form

$$\sum_{k=0}^{\infty} a r^k$$

This series converges if ____ $|r| < 1$ ____ and this series diverges if

____ $|r| \geq 1$ ____.

4. (5 pts.) Consider the definite integral below. Write down the sum, S_4, used to approximate the value of the integral below if Simpson's Rule is used with n = 4. **Do not attempt to evaluate the sum. Be very careful.**

$$\int_1^3 x^{1/4}\,dx$$

$$S_4 = \frac{1}{6}\left((\tfrac{2}{2})^{1/4} + 4(\tfrac{3}{2})^{1/4} + 2(\tfrac{4}{2})^{1/4} + 4(\tfrac{5}{2})^{1/4} + (\tfrac{6}{2})^{1/4} \right)$$

since $\Delta x = \frac{1}{2}$, *and* $x_k = 1 + \frac{k}{2} = \frac{2+k}{2}$, *for* $k = 0,1,2,3,4.$ are the points of the regular partition we need.

5. (5 pts.) Make an appropriate u-substitution of the form $u = x^{1/n}$ or $u = (x + a)^{1/n}$ and then evaluate the integral.

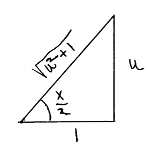
$$\int \frac{dx}{x(x - x^{1/4})} = \int \frac{4u^3}{u^4(1-u)}du = \int \frac{-4}{u(u-1)}du$$
$$= -4\int \frac{1}{u-1} - \frac{1}{u}\,du = \dots = 4\ln\left|\frac{x^{1/4}}{x^{1/4}-1}\right| + C$$

using the substitution $u = x^{1/4}$ so that $x = u^4$ and $dx = 4u^3\,du$.

6. (5 pts.) Evaluate the following integral using the substitution $u = \tan(x/2)$.

$$\int \frac{1}{1 - \sin(x)}\,dx = \int \frac{1}{1 - \left(\frac{2u}{1+u^2}\right)} \cdot \frac{2}{1+u^2}\,du = \int \frac{2}{(u-1)^2}\,du = -2(u-1)^{-1} + C$$
$$= \frac{2}{1-\tan(x/2)} + C$$

To use the substitution, from the triangle to the right, you may read off that $\sin(x/2) = u/(u^2 + 1)^{1/2}$ and that $\cos(x/2) = 1/(u^2 + 1)^{1/2}$. Then $\sin(x) = 2u/(u^2 + 1)$ and $\cos(x) = (1 - u^2)/(u^2 + 1)$ using double angle formulae. And finally, $u = \tan(x/2)$ implies $x = 2\tan^{-1}(u)$, and $dx = (2/(u^2 + 1))du$. Now put all this to work to evaluate the integral.



7. (10 pts.) Evaluate the integrals that converge.

(a)
$$\int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = \int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$$
$$= \lim_{a\to 1^+}\int_a^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \lim_{b\to\infty}\int_2^b \frac{1}{x\sqrt{x^2-1}}\,dx$$
$$= \lim_{a\to 1^+}[\sec^{-1}(2) - \sec^{-1}(a)] + \lim_{b\to\infty}[\sec^{-1}(b) - \sec^{-1}(2)]$$
$$= \dots = \frac{\pi}{2}$$

(b) $$\int_0^1 \frac{1}{(1-x)^{2/3}}\,dx = \lim_{b\to 1^-}\int_0^b (1-x)^{-2/3}\,dx = \dots = \lim_{b\to 1^-}[\,3 - 3(1-b)^{1/3}\,] = 3.$$

8. (4 pts.) Find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so, find its limit.

$$(\sqrt{2} - \sqrt{3}),\ (\sqrt{3} - \sqrt{4}),\ (\sqrt{4} - \sqrt{5}),\ \dots .$$

$a_n = \sqrt{n+1} - \sqrt{n+2}$ *for* $n \geq 1$. *Thus,* $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \dfrac{-1}{\sqrt{n+1} + \sqrt{n+2}} = 0.$

Observe we are dealing with a *sequence*. Note the commas. Note that no squeezing is needed, but it does provide an alternative legitimate route to the limit.

9. (4 pts.) Express the repeating decimal as a fraction.

$$0.141414\ldots = \frac{14}{99}$$

either by using the "high school" method or summing an appropriate geometric series.

10. (4 pts.) [Complete the following.] The harmonic series has the form

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

What is the sum of the harmonic series? The harmonic series does not converge, and thus, has no sum.

11. (8 pts.) Determine whether the series converges, and if so, find its sum.

(*a*) $\displaystyle\sum_{k=1}^{\infty} \left(-\frac{4}{3}\right)^{k-1}$ This is obviously a geometric series with $r = -4/3$.

Since $|r| \geq 1$, the series does not converge. No sum.

(*b*) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{(k+3)(k+4)} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2}{k+3} - \frac{2}{k+4} \right) = \lim_{n\to\infty} \left(\frac{2}{4} - \frac{2}{n+4} \right) = \frac{1}{2}.$

Note: Here the definition of the sum of an infinite series MUST be used.

12. (5 pts.) Find all values of x for which the series converges, and find the sum of the series for those values of x.

$$e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + e^{-5x} + \ldots$$

Evidently, this is a geometric series. Writing it using sigma notation makes things easy. Thus,

$$\sum_{k=1}^{\infty} e^{-kx} = \sum_{j=0}^{\infty} \left(e^{-x}\right)\left(e^{-x}\right)^{j} = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^{x} - 1}$$

provided that we have

$$\left|e^{-x}\right| < 1, \quad or \quad e^{-x} < 1\ , \quad or \quad 0 < x.$$

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13. (4 pts.) Use ratio test to determine whether the series converges, If the test is inconclusive, say so.

$\sum_{k=1}^{\infty} \frac{k!}{k^3}$ Since $\rho = \lim_{k\to\infty} \frac{(k+1)!/(k+1)^3}{k!/k^3} = \lim_{k\to\infty} \frac{k^3}{(k+1)^2} = \infty$, and $\rho > 1$, ratio test implies that the given series diverges.

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14. (4 pts.) Use root test to determine whether the series converges. If the test is inconclusive, say so.

$\sum_{k=1}^{\infty} \frac{k}{5^k}$ Since $\rho = \lim_{k\to\infty} \left[\left(\frac{k}{5^k}\right)\right]^{1/k} = \lim_{k\to\infty} \frac{k^{1/k}}{5} = \frac{1}{5}$, and $\rho < 1$, root test implies that the given series converges.

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15. (4 pts.) Use comparison test to show the following series converges.

$\sum_{k=1}^{\infty} \frac{5\sin^2(k)}{k^2}$ Plainly, for $k \geq 1$ we have $0 < 5\cdot\sin^2(k)/k^2 \leq 5/k^2$. Since the p-series $\sum k^{-2}$ converges, the series $\sum(5k^{-2})$ converges. Comparison against this last series now by using the inequality at the beginning of the paragraph allows us to conclude that the given series converges.

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16. (4 pts.) Apply the divergence test and state what it tells you about each of the following series.

(*a*) $\sum_{k=1}^{\infty} \frac{1}{k!}$ Since $\lim_{k\to\infty} \frac{1}{k!} = 0$,
divergence test provides no information concerning the convergence of (a).

(*b*) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k}+3}$ Since $\lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k}+3} = \lim_{k\to\infty} \frac{1}{1+\frac{3}{\sqrt{k}}} = 1$,

the limit of the sequence of terms is not equal to zero. Thus, divergence test implies that (b) diverges.

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17. (4 pts.) Confirm that the integral test is applicable and then use it to determine whether thefollowing series converges:

$\sum_{k=1}^{\infty} \frac{k}{1+k^2}$ Let $f(x) = x/(1+x^2)$ for $x \geq 1$. Clearly f is a positive continuous function. Since $f'(x) = (1-x^2)/(1+x^2)^2 < 0$ for $x > 1$, f is decreasing on $[1,\infty)$. Evidently the terms of the series are given by $f(k)$. Thus we may apply integral test.
Since we have

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{x}{1+x^2}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_1^{b} \frac{2x}{1+x^2}\,dx$$
$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln(1+b^2) - \frac{1}{2}\ln(2)\right] = \infty,$$

integral test implies that the given series diverges.

18. (6 pts.) Consider the sequence

$$a_1 = \sqrt{6}$$

$$a_2 = \sqrt{6 + \sqrt{6}}$$

$$a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$$

$$a_4 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}$$

$$\vdots$$

(a) Find a recursion formula for a_{n+1}.

$$a_{n+1} = \sqrt{6 + a_n} \quad \text{for } n \ge 1.$$

(b) Assuming the sequence converges, find the limit, L.

$$L = \lim_{n \to \infty} a_{n+1} = \sqrt{6 + \lim_{n \to \infty} a_n} = \sqrt{6 + L}$$

implies that $L^2 - L - 6 = 0$, so that $L = 3$ or $L = -2$. Since the sequence is nonnegative, the limit, if it exists must also be nonnegative. Thus it is impossible for L to be -2. As a consequence, $L = 3$.

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19. (4 pts.) From the definition of a limit of sequence, we know there is a positive integer N so that if $n \ge N$, then

$$\left| \frac{5n}{n+3} - 5 \right| < .0008$$

since

$$\lim_{n \to \infty} \frac{5n}{n+3} = 5.$$

Find a positive integer N which works and prove it provides the desired error bound. [You don't have to get the best. Just obtain one the you can prove works, and then convince the skeptics.]

Scratch: For $n \ge 1$,

$$\begin{aligned} |(5n/(n+3)) - 5| < .0008 \quad &\Leftrightarrow \quad |-15/(n+3)| < 8 \cdot 10^{-4} \\ &\Leftrightarrow \quad 15/(n+3) < 8 \cdot 10^{-4} \\ &\Leftrightarrow \quad (1.875)10^4 - 3 < n \\ &\Leftrightarrow \quad 18747 < n \quad /////// \end{aligned}$$

Let N = 18748. If N = 18748, then if $n \ge N$, then $n > 18747$, which is equivalent to $|(5n/(n+3)) - 5| < .0008$ from the logical equivalences in our scratch work. All you have to do is take the path from bottom to top.

Silly 10 Point Bonus: Prove that the sequence $\{a_n\}$ of Problem 18 actually converges. Say where your work is, for it won't fit here.