18. (6 pts.) Consider the sequence

$$a_1 = \sqrt{6}$$

$$a_2 = \sqrt{6 + \sqrt{6}}$$

$$a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$$

$$a_4 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}$$

$$\vdots$$

(a) Find a recursion formula for a_{n+1}.

(**) $$a_{n+1} = \sqrt{6 + a_n} \quad for\ n \geq 1.$$

(b) Assuming the sequence converges, find the limit, L.

$$L = \lim_{n \to \infty} a_{n+1} = \sqrt{6 + \lim_{n \to \infty} a_n} = \sqrt{6 + L}$$

implies that $L^2 - L - 6 = 0$, so that $L = 3$ or $L = -2$. Since the sequence is nonnegative, the limit, if it exists must also be nonnegative. Thus it is impossible for L to be -2. As a consequence, $L = 3$.

Silly 10 Point Bonus: Prove that the sequence $\{a_n\}$ of Problem 18 actually converges.

To prove that the sequence converges, we'll utilize the theorem that tells us that sequences that are bounded above and eventually increasing are convergent. To do this, using the recursive definition of the sequence via formula (**) above together with the needed basis, namely, that

(*) $$a_1 = \sqrt{6} ,$$

we shall show that the following three assertions are true by means of induction arguemnts:

(a) $a_n > 1$ *for all integers* $n \geq 1$;

(b) $a_n < 3$ *for all integers* $n \geq 1$; and

(c) $a_n < a_{n+1}$ *for all integers* $n \geq 1$.

You should note that (*) and (**) together constitute a recursive definition for the sequence. Assertions (b) and (c) provide us with the satisfaction of the essential hypotheses of the above mumbled about theorem.

Proof of (a): [Induction]

Basis Step: Since $a_1 = 6^{1/2} > 1^{1/2} = 1$, the basis step for the induction is satisfied.

Induction Step: Suppose n is an arbitrary positive integer and that $a_n > 1$. Using this, (**) and the observation that square roots preserve order, we have

$$a_{n+1} = \sqrt{6 + a_n} > \sqrt{6 + 1} > \sqrt{1} = 1.$$

Thus, for every positive integer n, $a_n > 1$ implies $a_{n+1} > 1$.

Having now satisfied the hypotheses of the induction axiom, we may now conclude that (a) is true.

Proof of (b): [Induction]

Basis Step: Since $a_1 = 6^{1/2} < 9^{1/2} = 3$, the basis step for the induction is satisfied.

Induction Step: Suppose n is an arbitrary positive integer and that $a_n < 3$. Using this, (**) and the observation that square roots preserve order, we have

$$a_{n+1} = \sqrt{6 + a_n} < \sqrt{6 + 3} = \sqrt{9} = 3.$$

Thus, for every positive integer n, $a_n < 3$ implies $a_{n+1} < 3$.

Having now satisfied the hypotheses of the induction axiom, we may now conclude that (b) is true.

As you can see, the induction arguments for (a) and (b) are quite similar. One might, in fact, say they are the same *mutatis mutandis*. [If you don't know the meaning of that phrase, you might want to break out your pet dictionary or Google --- whichever is the most convenient.]

Proof of (c): [Induction]

Basis Step: Using (a) with n = 1, (**), and the observation that square roots preserve order, we see easily that

$$a_1 = 6^{1/2} < (6 + 1)^{1/2} < (6 + a_1)^{1/2} = a_{1+1}.$$

Consequently, the basis step for the induction is satisfied.

[Note that $1 < a_1$ is not really needed, for $0 < a_1$ suffices here.]

Induction Step: Suppose n is an arbitrary positive integer and that $a_n < a_{n+1}$. Using this, (**) and the observation that square roots preserve order, we have

$$a_{n+1} = \sqrt{6 + a_n} < \sqrt{6 + a_{n+1}} = a_{(n+1)+1}.$$

Thus, for every positive integer n, $a_n < a_{n+1}$ implies $a_{n+1} < a_{(n+1)+1}$.

Having now satisfied the hypotheses of the induction axiom, we may now conclude that (c) is true.

And just what is that *induction axiom*?? It is a codification of one of the most important properties of the set of positive integers. When formulated in terms of sets, it looks like this:

Let $\mathbb{N}^+ = \{1, 2, 3, \dots\}$ denote the set of positive integers. Suppose

$A \subseteq \mathbb{N}^+$. If

$1 \in A$, (basis step)

and

for every positive integer n, $n \in A \Rightarrow n+1 \in A$, (induction step),

then $A = \mathbb{N}^+$.

The arguments proving (a) - (c) above are informal, but may be cast formally in set theoretic language, if desired. e.t.