READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds. Eschew obfuscation.*

1. (5 pts.) Using a complete sentence and appropriate notation, state precisely the First Part of the Fundamental Theorem of Calculus. [This is sometimes called the *Evaluation Theorem*.]

If $f(x)$ is continuous on $[a,b]$ and $g(x)$ is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\ dx = g(b) - g(a).$$

2. (5 pts.) Evaluate the following using the 1st Part of the Fundamental Theorem of Calculus.

$$\int_{-\pi/4}^{\pi/4} 2\cos(2x)\ dx = \sin(2x)\Big|_{-\pi/4}^{\pi/4} = \sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2}) = 2.$$

Note: One might also explicitly do a u-substitution here.

3. (5 pts.) Express the following sum using sigma notation, but do not attempt to find its numerical value.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} = \sum_{k=1}^{7} \frac{(-1)^{k+1}}{2k-1}$$

Observe that this equation may be completed in infinitely many correct ways. You need only re-index to change the form of the varmint.

4. (5 pts.) Obtain the upper limit of summation and rewrite the summand in order to make the following equation true. **Do not attempt to evaluate the sum.**

$$\sum_{k=10}^{45} k^4 = \sum_{j=1}^{36} (j+9)^4$$

since $j = k - 9$ is equivalent to $k = j + 9$.

5. (5 pts.) Express the following limit as a definite integral. Do not attempt to evaluate the integral.

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (2\cos^2 x_k^*)\Delta x_k \ ; \quad a = -\pi/2,\ b = \pi/2.$$

$$L = \int_{-\pi/2}^{\pi/2} 2\cos^2(x)\ dx$$

Silly 10 Point Bonus: Obtain the exact numerical value of the limit L in Problem 5 on Page 1 of 4.

$$L = \int_{-\pi/2}^{\pi/2} 2\cos^2(x)\ dx = \int_{-\pi/2}^{\pi/2} 1 + \cos(2x)\ dx = \left(x + \frac{\sin(2x)}{2}\right)\Bigg|_{-\pi/2}^{\pi/2} = \ldots = \pi$$

Trig or treat!! [Integration by parts is less accessible at this time.]

6. (5 pts.) Using complete sentences and appropriate notation, state the Second Part of the Fundamental Theorem of Calculus.

// Let f(x) be a function that is continuous on an interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\, dt,$$

for each x in I, then g′(x) = f(x) for each x in I.//

7. (10 pts.) Find each of the following derivatives.

(a) $\frac{d}{dx}\left[\int_{\sqrt{3}}^{x} \tan^{-1}(t)\, dt \right] = \tan^{-1}(x)$

(b) $\frac{d}{dx}\left[\int_{1}^{\ln(x)} e^{t}\, dt \right] = (e^{\ln(x)})\left(\frac{1}{x}\right) = \frac{x}{x} = 1$ *for* $x > 0$.

8. (5 pts.) (a) (3 pts.) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: To start, complete the sentence,"ln(x) = ...")

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

for $x > 0$. The domain of the natural log function is $(0,\infty)$, and its range is the whole real line, $(-\infty,\infty)$.

(b) (2 pts.) Given ln(a) = 3 and ln(b) = -2, evaluate the following integral.

$\int_1^{ab^2} \frac{1}{t}\, dt = \ln(ab^2) = \ln(a) + \ln(b^2) = \ln(a) + 2\ln(b) = 3 + 2(-2) = -1.$

9. (5 pts.) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable **t**, so the differential denoting the variable of integration is ***dt***. DO NOT ATTEMPT TO EVALUATE THE DEFINITE INTEGRAL IN t THAT YOU OBTAIN.

$$\frac{dy}{dx} = \sec^2(x) - \sin(x), \quad y(\pi/4) = 1.$$

$y(x) = \int_{\pi/4}^{x} \sec^2(t) - \sin(t)\, dt + 1$

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10. (5 pts.) Evaluate the definite integral below by expressing it in terms of u correctly and then evaluating the resulting integral using a formula from geometry.

$$\int_{e^{-3}}^{e^{3}} \frac{\sqrt{9-(\ln(x))^2}}{x}\,dx \ ; \ u = \ln(x)$$

$$\int_{e^{-3}}^{e^{3}} \frac{\sqrt{9-(\ln(x))^2}}{x}\,dx = \int_{-3}^{3} \sqrt{9-u^2}\,du = \frac{9\pi}{2}$$

since $du = 1/x\ dx$. The right-most integral provides the area of half a circle with a radius of 3 centered at the origin.

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11. (10 pts.) Write each of the following two sums in closed form.

(*a*) $\displaystyle\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{1} - \frac{1}{51} = \frac{50}{51}$

(*b*) $\displaystyle\sum_{k=0}^{100}(-1)^{k+1}\left(\frac{1}{2^k}\right) = \sum_{k=0}^{100}(-1)\left(-\frac{1}{2}\right)^k = \ \ldots\ = \frac{2}{3}\left[(-1/2)^{101} - 1\right]$

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12. (10 pts.) If the function f is continuous on [a,b], then the *net signed area* A between $y = f(x)$ and the interval [a,b] is defined by

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Reveal all the details in computing the numerical value of the net signed area of $f(x) = x^3$ over the interval [0,2] using only the definition above with

$$x_k^*$$

the right end point of each subinterval in the regular partition.

$$\begin{aligned} A &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2k}{n}\right)^3\frac{2}{n} \\ &= \lim_{n\to\infty}\left(\frac{16}{n^4}\right)\sum_{k=1}^{n}k^3 = \lim_{n\to\infty}\left(\frac{16n^2(n+1)^2}{4n^4}\right) \\ &= \lim_{n\to\infty} 4\left(1+\frac{1}{n}\right)^2 = 4 \end{aligned}$$

since $\Delta x = 2/n$ and $x_k = 2k/n$ for $k = 0, 1, \ldots, n$ are the end points of the intervals of the general regular partition.

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13. (10 pts.) A particle moves with a velocity of $v(t) = \cos(t)$ along an s-axis. Find the displacement and total distance traveled over the time interval $[0, 3\pi/2]$.

$$\text{Displacement} = \int_0^{3\pi/2} v(t)\ dt = \int_0^{3\pi/2} \cos(t)\ dt = \ldots = -1$$

$$\text{Total\_Distance} = \int_0^{3\pi/2} |v(t)|\ dt = \int_0^{3\pi/2} |\cos(t)|\ dt = \ldots = 3$$

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14. (5 pts.) Evaluate the following limit:

$$L = \lim_{x \to 0} (1 + 2x)^{\frac{1}{x}} = \lim_{u \to 0} (1 + u)^{\frac{2}{u}} = e^2$$

using the substitution $u = 2x$.

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15. (10 pts.) Let the function g be defined by the equation

$$g(x) = \int_0^x \cos^4(t)\ dt - 3x$$

for $x \in (-\pi/2, \pi/2)$. Then since

$$g'(x) = \cos^4(x) - 3 \quad \text{and} \quad g''(x) = -4\sin(x)\cos^3(x) \ ,$$

(a) $g(0) = \int_0^0 \cos^4(t)\ dt - 3(0) = 0$

(b) $g'(0) = \cos^4(0) - 3 = -2$

(c) $g''(0) = -4\cos^3(0)\sin(0) = 0$

(d) Determine the open intervals where g is increasing or decreasing. Be specific.

Since $g'(x) \leq -2$ for $x \in (-\pi/2, \pi/2)$, g is decreasing on $(-\pi/2, \pi/2)$.

(e) Determine the open intervals where g is concave up or concave down. Be specific.

By considering the sign of g'' above, we can see that g is concave up on $(-\pi/2, 0)$ and concave down on $(0, \pi/2)$. Hint: cosine is positive on the interval $(-\pi/2, \pi/2)$. sin(x) is the determining factor. //

Silly 10 Point Bonus: Obtain the exact numerical value of the limit L in Problem 5 on Page 1 of 4. [Say where your work is, for it won't fit here.]

This may be found on the bottom of Page 1 of 4.