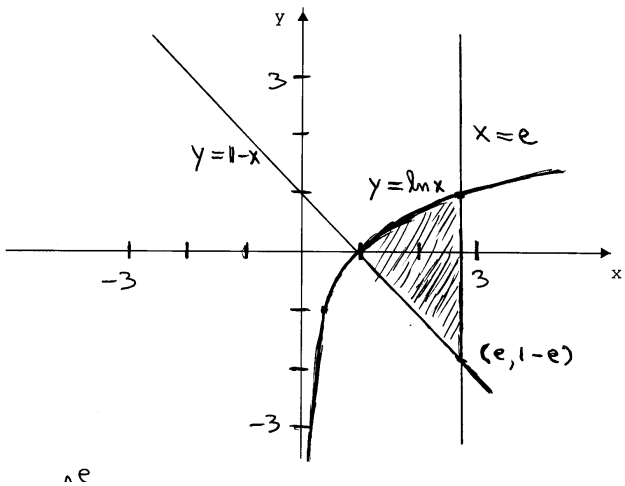
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (25 pts.) The region ~~in the first quadrant~~ enclosed by the curves $y = 1 - x$, $y = \ln x$ and $x = e$ is sketched below for your convenience. [*Corrected in Class.*]



(a) Write down, but do not attempt to evaluate the definite integral whose numerical value gives the area of the region R if one integrates with respect to x so the differential in the integral is dx.

$$\text{Area} = \int_1^e \ln(x) - (1-x)\ dx$$

(b) Write down, but do not attempt to evaluate the sum of definite integrals whose numerical value gives the area of the region R if one integrates with respect to y so the differential in the integral is dy.

$$\text{Area} = \int_{1-e}^{0} e - (1-y)\ dy + \int_0^1 e - e^y\ dy$$

(c) Using the method of cylindrical shells, write a single definite integral dx whose numerical value is the volume of the solid obtained when the region R above is revolved around the y-axis. Do not evaluate the integral.

$$\text{Volume} = \int_1^e 2\pi x(\ln(x) - (1-x))\ dx$$

(d) Using the method of disks or washers, write down a sum of definite integrals dy to compute the same volume as in part (c). Do not evaluate the integrals.

$$\text{Volume} = \int_{1-e}^{0} \pi\,(e^2 - (1-y)^2)\ dy + \int_0^1 \pi\,(e^2 - e^{2y})\ dy$$

(e) Write down, but do not attempt to evaluate, the definite integral that gives the arc-length of the curve $y = \ln(x)$ from $x = 1$ to $x = e$.

$$\text{Length} = \int_1^e \sqrt{1 + (dy/dx)^2}\ dx = \int_1^e \sqrt{1 + \left(\frac{1}{x}\right)^2}\ dx$$

$$= \int_1^e \frac{\sqrt{x^2+1}}{x}\ dx$$

or

$$\text{Length} = \int_0^1 \sqrt{1 + (dx/dy)^2}\ dy = \int_0^1 \sqrt{1 + (e^y)^2}\ dy$$

$$= \int_0^1 \sqrt{1 + e^{2y}}\ dy$$

2. (15 pts.) (a) (10 pts.) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc. Be very careful here.

$$\frac{4x^2+5}{x(x+1)^2(4x^2+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{4x^2+1} + \frac{Fx+G}{(4x^2+1)^2}$$

(b) (5 pts.) If one were to integrate the rational function in part (a), one probably would encounter the integral below. Reveal, in detail, how to evaluate this integral.

$$\int \frac{dx}{(4x^2+1)^2} = \int \frac{dx}{((2x)^2+1)^2} = \int \frac{\sec^2(\theta)\,d\theta}{2(\sec^2(\theta))^2},\ \textit{when}\ \tan(\theta) = 2x,$$

$$= \int \frac{\cos^2(\theta)}{2}\,d\theta = \int \frac{1+\cos(2\theta)}{4}\,d\theta = \frac{\theta}{4} + \frac{\sin(\theta)\cos(\theta)}{4} + C$$

$$= \frac{\tan^{-1}(2x)}{4} + \frac{2x}{4(4x^2+1)} + C.$$

An appropriately drawn right triangle might be a big help at the end.

3. (60 pts.) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[**6 pts./part**]

(a) $\int 4x\cos(2x)\,dx = 2x\sin(2x) - \int 2\sin(2x)\,dx = 2x\sin(2x) + \cos(2x) + C$

by integrating by parts using $u = 2x$ and $dv = 2\cos(2x)dx$. One could, of course, do a u-substitution first to simplify things, and then integrate by parts, etc. We did some of the easier u-substitutions in the bio-computer.

(b)

$$\int_0^{(\pi/6)^{1/2}} 4x\sin(x^2)\,dx = \int_0^{\pi/6} 2\sin(u)\,du = (-2\cos(u))\Big|_0^{\pi/6}$$

$$= -2\cos(\pi/6) - (-2\cos(0)) = 2 - \sqrt{3}$$

using the u-substitution $u = x^2$.

(c) $\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + c$

by integrating by parts using $u = \sin^{-1}(x)$ and $dv = 1\cdot dx$, and then performing an obvious u-substitution.

(d)

$$\int 8x^2e^x\,dx = 8x^2e^x - \int 16xe^x\,dx = 8x^2e^x - \left(16xe^x - \int 16e^x\,dx\right)$$

$$= 8x^2e^x - 16xe^x + 16e^x + C$$

by integrating by parts twice in succession, all the while continually picking on our really beloved exponential function as the recognized derivative.

3. (Continued) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[**6 pts./part**]

(e)

$$\int \frac{2}{\sin(t)\cos(t)}\,dt = \int \frac{4}{\sin(2t)}\,dt = \int 4\,csc(2t)\,dt$$

$$= -2\ln\left|csc(2t) + \cot(2t)\right| + C$$

After the trig-magic there is an obvious and easy u-substitution to be done.

(f)

$$\int \sqrt{4 - x^2}\,dx = 2\int \sqrt{1 - \left(\frac{x}{2}\right)^2}\,dx$$

$$= \int 4\cos^2(\theta)\,d\theta = \int 2 + 2\cos(2\theta)\,d\theta$$

$$= 2\theta + \sin(2\theta) + C = 2\theta + 2\sin(\theta)\cos(\theta) + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4 - x^2}}{2} + C$$

using the obvious trigonometric substitution $x/2 = \sin(\theta)$. Of course you could also try integration by parts, but that route is a little thornier.

(g)

$$\int \frac{2x^4 + 4x^3}{x^2 + 1}\,dx = \int 2x^2 + 4x - 2 - \frac{4x - 2}{x^2 + 1}\,dx$$

$$= \frac{2}{3}x^3 + 2x^2 - 2x - 2\ln\left|x^2 + 1\right| + 2\tan^{-1}(x) + C$$

after doing an easy long division.

(h) $$\int \sec(4t)\,dt = \frac{1}{4}\ln\left|\sec(4t) + \tan(4t)\right| + C$$

(i) $$\int \frac{12}{x^2 - 9}\,dx = \int \frac{2}{x - 3}\,dx - \int \frac{2}{x + 3}\,dx = 2\ln\left|\frac{x - 3}{x + 3}\right| + C.$$

after performing an easy partial-fraction decomposition.

(j)

$$\int \cos(x)e^x\,dx = \cos(x)e^x - \int (-\sin(x))e^x\,dx$$

$$= \cos(x)e^x + \left(\sin(x)e^x - \int \cos(x)e^x dx\right)$$

by integrating by parts twice in succession, all the while continually picking on our beloved exponential function as the recognized derivative. Solving this little linear equation allows us to write

$$\int \cos(x)e^x\,dx = \left(\frac{\sin(x) + \cos(x)}{2}\right)e^x + C.$$

Note: Integrals (e) and (i) can be obtained by alternative, more lengthy routes. For (i), the trigonometric substitution $x = 3\sec(\theta)$ and much careful work will do the trick. For (e), one can begin by multiplying the numerator and denominator of the integrand by $\cos(t)$. If this is followed by pythagorean magic, a u-substitution, and a partial-fraction decomposition, then you will get a loggy mess equivalent to what we provided.

Silly 10 Point Bonus: State and prove the Mean-Value Theorem of Integrals. In doing this, feel free to use the Extreme-Value Theorem and the Intermediate-Value Theorems, which codify important behaviors of continuous functions. //Say where your work is, for there isn't room here.

Mean-Value Theorem of Integrals. If f is a continuous function defined on a nontrivial closed interval $[a,b]$, then there is a number $x^* \in [a,b]$ with

$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

To prove this using the Extreme-Value Theorem and the Intermediate-Value Theorems, we proceed as follows:

First, assuming f is a continuous function defined on a nontrivial closed interval $[a,b]$, the Extreme-Value Theorem implies that f had a maximum and a minimum on the interval. This actually means that there are numbers m and M with $m \le M$ and numbers x_0 and x_1 in the interval with $f(x_0) = m$, $f(x_1) = M$, and $m \le f(x) \le M$ for each $x \in [a,b]$. From properties of the definite integral, since f is continuous, not only is it integrable on the interval, but we then have

$$\int_a^b m\, dx \le \int_a^b f(x)\, dx \le \int_a^b M\, dx$$

which implies

$$m(b-a) \le \int_a^b f(x)\, dx \le M(b-a) \ .$$

Consequently, multiplying through by the reciprocal of $b - a$ yields

$$m \le \frac{1}{b-a}\int_a^b f(x)\, dx \le M \ .$$

To finish things off, we have only to observe that since f is continuous on the interval $[a,b]$ and the number

$$\frac{1}{b-a}\int_a^b f(x)\, dx$$

lies in the closed interval $[m,M]$ whose boundary points are the minimum and maximum values of f on the interval $[a,b]$, the Intermediate-Value Theorem implies that there is an $x^* \in [a,b]$ with

$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)\, dx.$$