

READ ME FIRST: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals", " \Rightarrow " denotes "implies", and " \Leftarrow " denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page. Eschew obfuscation.

1. (10 pts.) Consider the definite integral below. (a) Write down the sum, S_6 , used to approximate the value of the integral below if Simpson's Rule is used with $n = 6$. Do not attempt to evaluate the sum. (b) Write down the sum, T_6 , used to approximate the value of the integral below if Trapezoid Rule is used with $n = 6$. Do not attempt to evaluate the sum.

$$\int_0^3 \sqrt{x} \, dx$$

Plainly, $\Delta x = \frac{1}{2}$, and $x_k = 0 + \frac{k}{2} = \frac{k}{2}$, for $k = 0, 1, 2, \dots, 6$

are the points of the regular partition we need.

$$\begin{aligned} (a) \quad S_6 &= \frac{1}{3} \Delta x (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{1}{6} \left(\left(\frac{0}{2}\right)^{1/2} + 4\left(\frac{1}{2}\right)^{1/2} + 2\left(\frac{2}{2}\right)^{1/2} + 4\left(\frac{3}{2}\right)^{1/2} + 2\left(\frac{4}{2}\right)^{1/2} + 4\left(\frac{5}{2}\right)^{1/2} + \left(\frac{6}{2}\right)^{1/2} \right) \end{aligned}$$

$$\begin{aligned} (b) \quad T_6 &= \frac{1}{2} \Delta x (y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6) \\ &= \frac{1}{4} \left(\left(\frac{0}{2}\right)^{1/2} + 2\left(\frac{1}{2}\right)^{1/2} + 2\left(\frac{2}{2}\right)^{1/2} + 2\left(\frac{3}{2}\right)^{1/2} + 2\left(\frac{4}{2}\right)^{1/2} + 2\left(\frac{5}{2}\right)^{1/2} + \left(\frac{6}{2}\right)^{1/2} \right) \end{aligned}$$

2. (10 pts.) Evaluate the integrals that converge.

$$(a) \quad \int_{-\infty}^3 \frac{2x \, dx}{x^2 + 9} = \lim_{a \rightarrow -\infty} \int_a^3 \frac{2x \, dx}{x^2 + 9} = \lim_{a \rightarrow -\infty} (\ln(18) - \ln(a^2 + 9)) = -\infty.$$

As a consequence, this improper integral diverges.

$$\begin{aligned} (b) \quad \int_{\frac{1}{2}}^1 \frac{2}{\sqrt{1-x^2}} \, dx &= \lim_{b \rightarrow 1^-} \int_{\frac{1}{2}}^b \frac{2}{\sqrt{1-x^2}} \, dx \\ &= \lim_{b \rightarrow 1^-} \left(2 \sin^{-1}(b) - 2 \sin^{-1}\left(\frac{1}{2}\right) \right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \end{aligned}$$

Silly 10 Point Bonus: Prove that

$$\sum_{k=1}^n \frac{1}{k^4} < \frac{4}{3}$$

for every $n \geq 1$. /// First observe that the inequality is true if $n = 1$, and

$$\frac{1}{(k+1)^4} < \int_k^{k+1} x^{-4} \, dx = \frac{1}{3k^3} - \frac{1}{3(k+1)^3}$$

for $k \geq 1$. [Think proof of integral test!!!] Consequently, if $n \geq 2$,

$$\sum_{k=1}^n \frac{1}{k^4} = 1 + \sum_{k=1}^{n-1} \frac{1}{(k+1)^4} \leq 1 + \sum_{k=1}^{n-1} \left(\frac{1}{3k^3} - \frac{1}{3(k+1)^3} \right) = 1 + \frac{1}{3} - \frac{1}{3(n)^3} < \frac{4}{3}.$$

3. (4 pts.) Express the repeating decimal as a fraction, more specifically as a quotient of positive integers. [The fraction does not have to be in lowest terms.]

$$0.75757575 \dots = \frac{75}{99} = \frac{25}{33}$$

either by using the "high school" method or summing an appropriate geometric series.

4. (4 pts.) Find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so, find its limit.

$$\frac{1}{3}, -\frac{2}{5}, \frac{3}{7}, -\frac{4}{9} \dots$$

$$a_n = (-1)^{n+1} \left(\frac{n}{2n+1} \right) \text{ for } n \geq 1. \text{ Thus, } \lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}.$$

Since the sequence is alternating in sign, this means the limit does not exist. In fact, the sequence *clusters* at $1/2$ and $-1/2$.

5. (4 pts.) [Complete the following.] The harmonic series has the form

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

What is the sum of the harmonic series? The harmonic series does not converge, and thus, has no sum.

6. (8 pts.) Determine whether the series converges, and if so, find its sum.

$$(a) \quad \sum_{k=1}^{\infty} \left(\frac{2}{3} \right)^{k+2} = \sum_{j=0}^{\infty} \left(\frac{2}{3} \right)^{j+3} = \sum_{j=0}^{\infty} \left(\frac{2}{3} \right)^3 \left(\frac{2}{3} \right)^j = \left(\frac{8}{27} \right) \left(\frac{1}{1 - (2/3)} \right) = \frac{8}{9}$$

This is obviously a geometric series with $r = 2/3$. Since $|r| < 1$, the series converges and the computations are easy.

$$(b) \quad \sum_{k=1}^{\infty} \left(\frac{10}{k+4} - \frac{10}{k+5} \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{10}{k+4} - \frac{10}{k+5} \right) \\ = \lim_{n \rightarrow \infty} \left(\frac{10}{1+4} - \frac{10}{n+5} \right) = 2$$

Note: Here the definition of the sum of an infinite series MUST be used.

7. (4 pts.) Use root test to determine whether the series converges. If the test is inconclusive, say so.

$$\sum_{k=1}^{\infty} \left(\frac{3k+2}{2k-1} \right)^k \quad \text{Since } \rho = \lim_{k \rightarrow \infty} \left[\left(\frac{3k+2}{2k-1} \right)^k \right]^{1/k} = \lim_{k \rightarrow \infty} \frac{3k+2}{2k-1} = \frac{3}{2} > 1, \quad \text{root test}$$

implies that the given series diverges.

8. (4 pts.) Apply the divergence test and state what it tells you about each of the following series.

(a) $\sum_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^k$ Since $\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k = e \neq 0$, divergence test implies that (a) diverges. [You may also observe that the sequence of terms is bounded away from zero in an appropriate way.]

(b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ Since $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{k}} = 0$, divergence test provides no information concerning the convergence of (b).

9. (4 pts.) Use ratio test to determine whether the series converges. If the test is inconclusive, say so.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{Since } \rho = \lim_{k \rightarrow \infty} \frac{1/(k+1)^2}{1/k^2} = \lim_{k \rightarrow \infty} \frac{k^2}{(k+1)^2} = \dots = 1, \quad \text{ratio test is}$$

inconclusive. [Note: Although the series actually converges, ratio test doesn't tell you anything about it.]

10. (4 pts.) Use comparison test to show the following series diverges.

$$\sum_{k=1}^{\infty} \frac{\ln(k)}{k} \quad \text{First, } 1/k \leq \ln(k)/k \text{ for } k \geq 3. \text{ Since the p-series } \sum k^{-1}$$

diverges, its tail series $\sum_{k=3}^{\infty} (1/k)$ also diverges. From

comparison test, now, it follows that $\sum_{k=3}^{\infty} (\ln(k)/k)$ diverges. Finally, since a tail series of #10 diverges, the series of problem #10 diverges.

11. (4 pts.) Confirm that the integral test is applicable, and then use it to determine whether the following series converges:

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2} \quad \text{Let } f(x) = 1/(1+x^2) \text{ for } x \geq 1. \text{ Plainly } f \text{ is a positive}$$

continuous function, and $1/(1+k^2) = f(k)$ for $k \geq 1$. Since

$f'(x) = -2x/(1+x^2)^2 < 0$ for $x > 1$, it follows that f is decreasing for $x \geq 1$. This means that we may use f , defined above, in integral test to determine whether the given series converges. Since

$$\int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} (\tan^{-1}(b) - \tan^{-1}(1)) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

it follow from integral test that the given series converges.

12. (4 pts.) Find all values of x for which the series converges, and find the sum of the series for those values of x .

$$\frac{1}{x^2} + \frac{2}{x^3} + \frac{4}{x^4} + \frac{8}{x^5} + \frac{16}{x^6} + \dots$$

Evidently, this is a geometric series. Writing the series using sigma notation makes things easy. Thus,

$$\sum_{k=0}^{\infty} \frac{2^k}{x^{k+2}} = \sum_{k=0}^{\infty} \left(\frac{1}{x^2} \right) \left(\frac{2}{x} \right)^k = \left(\frac{1}{x^2} \right) \left(\frac{1}{1 - \left(\frac{2}{x} \right)} \right) = \frac{1}{x^2 - 2x}$$

provided that we have

$$\left| \frac{2}{x} \right| < 1, \quad \text{or} \quad 2 < |x|, \quad \text{or} \quad (x < -2 \text{ or } 2 < x).$$

13. (6 pts.) [Complete the following.] A geometric series is a series of the form

$$\sum_{k=0}^{\infty} ar^k$$

This series diverges if $|r| \geq 1$ and this series converges if $|r| < 1$.

14. (4 pts.) Suppose that

$$\sum_{n=1}^{\infty} a_n = \frac{2}{5} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = -6.$$

Then we have

$$\sum_{n=1}^{\infty} \left(60a_n + \frac{5}{3}b_n \right) = 60 \sum_{n=1}^{\infty} a_n + \frac{5}{3} \sum_{n=1}^{\infty} b_n = (60) \left(\frac{2}{5} \right) + \left(\frac{5}{3} \right) (-6) = 24 - 10 = 14.$$

15. (6 pts.) [Complete the following.] A p-series is a series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

This series diverges if $p \leq 1$ and this series converges if $p > 1$.

16. (8 pts.) (a) Using complete sentences and appropriate notation, give the precise ε - N definition of

$$(*) \quad \lim_{n \rightarrow \infty} a_n = L.$$

We write (*) above if L is a number such that, for each $\varepsilon > 0$, there is a positive integer N , dependent on ε , such that for every positive integer n , if $n \geq N$, then $|a_n - L| < \varepsilon$.

(b) Give the precise mathematical definition of the sum of an infinite series,

$$(**) \quad \sum_{k=1}^{\infty} a_k$$

A number s is the sum of the series (**) above if

$$s = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

If the limit fails to exist, the series is said to diverge.

17. (4 pts.) From the definition of a limit of sequence, we know there is a positive integer N so that if $n \geq N$, then

$$(*) \quad \left| \frac{n}{n+1} - 1 \right| < \left(\frac{1}{2} \right) 10^{-3} \quad \text{since} \quad \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1.$$

Find a positive integer N which works and prove it provides the desired error bound. // By doing a little routine algebra, it is easy to see that when $n \geq 1$, inequality (*) above is equivalent to

$$\left| \frac{-1}{n+1} \right| < \left(\frac{1}{2} \right) 10^{-3} \Leftrightarrow \frac{1}{n+1} < \left(\frac{1}{2} \right) 10^{-3} \Leftrightarrow 2000 < n+1 \Leftrightarrow 1999 < n.$$

It follows that if we let $N = 2000$, then if $n \geq N$, then $n > 1999$, which is equivalent to (*). Just trace the double-headed arrow path backwards.

18. (8 pts.) Let the sequence $\{a_n\}$ be defined recursively by

$$a_1 = \sqrt{2}, \text{ and } a_{n+1} = \sqrt{2 + a_n}$$

for $n \geq 1$. (a) List the first four terms of the sequence.

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2 + \sqrt{2}}, \quad a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \quad \text{and} \quad a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

(b) Assuming the sequence converges, find its limit L .

$$L = \lim_{n \rightarrow \infty} a_{n+1} = \sqrt{2 + \lim_{n \rightarrow \infty} a_n} = \sqrt{2 + L}$$

implies that $L^2 - L - 2 = 0$, so that $L = 2$ or $L = -1$. Since the sequence is nonnegative, the limit, if it exists must also be nonnegative. Thus it is impossible for L to be -1 . As a consequence, $L = 2$.

Silly 10 Point Bonus: Prove that

$$\sum_{k=1}^n \frac{1}{k^4} < \frac{4}{3}$$

for every $n \geq 1$. [Say where your work is, for it won't fit here. **Page 1.**]