NAME: eM toidI

STUDENT NUMBER: 0000000 EXAM NUMBER: 00

Read Me First:

Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Remember that what is illegible or incomprehensible is worthless. Communicate. Eschew obfuscation. Good Luck! [Total Points Possible: 120]

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1. (20 pts.) (a) Using a complete sentence, state the first part of the Fundamental Theorem of Calculus, the *evaluation* theorem.

// If $f(x)$ is continuous on $[a,b]$ and $g(x)$ is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\ dx = g(b) - g(a).$$

(b) Using complete sentences, state the second part of the Fundamental Theorem of Calculus.

// Let f(x) be a function that is continuous on an interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\ dt,$$

for each x in I, then $g'(x) = f(x)$ for each x in I.//

(c) Compute $g'(x)$ when $g(x)$ is defined by the following equation.

$$g(x) = \int_0^x e^{t^2} + 5\ dt + \sin^{-1}(x)$$

$$g'(x) = e^{x^2} + 5 + \frac{1}{\sqrt{1 - x^2}}$$

(d) Give the definition of the function ln(x) in terms of a definite integral, and give its domain and range. Label correctly.

$$\ln(x) = \int_1^x \frac{1}{t}\ dt$$

for $x > 0$. The domain of the natural log function is $(0,\infty)$, and its range is the whole real line, $(-\infty,\infty)$.

(e) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable t, so the differential denoting the variable of integration is dt. Then reveal an alternative identity of y by actually evaluating the definite integral with respect to t.

$$\frac{dy}{dx} = \frac{5\ln^4(x)}{x} \quad \text{with} \quad y(e) = 6.$$

$$y(x) = 6 + \int_e^x \frac{5\ln^4(t)}{t}\ dt$$

$$= 6 + \ln^5(x) - \ln^5(e) = \ln^5(x) + 5.$$

2. (5 pts.) Suppose that

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2 5^k}(x-3)^k$$

Find the radius of convergence and the interval of convergence of *f*.

// To use ratio test for absolute convergence, we compute

$$\rho(x) = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2\frac{1}{5}|x-3| = \frac{1}{5}|x-3|.$$

Plainly,

$$\rho(x) < 1 \quad\Leftrightarrow\quad \frac{1}{5}|x-3| < 1 \quad\Leftrightarrow\quad |x-3| < 5.$$

Thus, the radius of convergence is R = 5. By unwrapping the rightmost inequality above, we can obtain the interior of the interval of convergence, namely, the interval (-2,8). Substitution of $x = -2$ into f yields

$$\sum_{k=1}^{\infty} \frac{-1}{k^2}$$

which converges. Also substitution of $x = 8$ into f yields

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$$

which also converges. The interval of convergence: I = [-2,8].

3. (10 pts.) Each of the following power series functions is the Maclaurin series of some well-known function. In each case, (i) identify the function, and (ii) provide the interval in which the series actually converges to the function.

(a) $$\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = \ln(1+x) \quad \textit{for } x \in (-1,1].$$

(b) $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \textit{for } x \in (-1,1).$$

(c) $$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \cos(x) \quad \textit{for } x \in \mathbb{R} = (-\infty,\infty).$$

(d) $$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = \tan^{-1}(x) \quad \textit{for } x \in [-1,1].$$

(e) $$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin(x) \quad \textit{for } x \in \mathbb{R} = (-\infty,\infty).$$

4. (5 pts.) Let $f(x) = \sin(\pi x)$.

Obtain the 3rd Taylor polynomial of $f(x)$ about $x_0 = 1$.

$$p_3(x) = f(1) + \frac{f^{(1)}(1)}{1!}(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= -\frac{\pi}{1!}(x-1) + \frac{\pi^3}{3!}(x-1)^3$$

5. (8 pts.) (a) $\ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$

Use the error estimate from alternating series test to determine a specific value of $n \geq 1$ so that the partial sum s_n approximates ln(2) to 5 decimal places, where, of course,

$$s_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$$

Since

$$|s_n - \ln(2)| < \frac{1}{n+1} \quad for \quad n \geq 1,$$

it suffices to find a positive integer n so that

$$\frac{1}{n+1} \leq \left(\frac{1}{2}\right)10^{-5}$$

is true. Solving this inequality for n and taking into account that n must be a positive integer yields $n \geq 199999$. Thus, take $n = 199999$. //

(b) Use the *Remainder Estimation Theorem* to obtain an interval containing $x = 0$ in which $f(x) = \cos(x)$ can be approximated to three decimal place accuracy by

$$p(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

Since $p(x) = p_5(x)$, the 5th Maclaurin polynomial of f, the 6th derivative of f is $-\cos(x)$, and $|\cos(x)| \leq 1$ for all x, we may use the Remainder Estimation Theorem to deduce that

$$|\cos(x) - p(x)| = |\cos(x) - p_5(x)| \leq \frac{|x|^6}{6!}$$

for every real number x. Thus, to obtain the desired accuracy, it suffices to ensure that

$$\frac{|x|^6}{6!} < \frac{1}{2}10^{-3} \quad \text{which is equivalent to} \quad |x| < (0.36)^{1/6}.$$

This means that an appropriate interval is $I = (-(.36)^{1/6}, (.36)^{1/6})$. //

6. (6 pts.) Classify each of the following series as absolutely convergent (AC), conditionally convergent (CC), divergent (D), or none of the preceding, (N). Circle the letters corresponding to your choice. (No explicit proof is needed.)

(a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+1}$ **(AC)** ~~(CC)~~ ~~(D)~~ ~~(N)~~

(b) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k!}{k^{50}+1}$ ~~(AC)~~ ~~(CC)~~ **(D)** ~~(N)~~

(c) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{4k}}$ ~~(AC)~~ **(CC)** ~~(D)~~ ~~(N)~~

7. (6 pts.) Here are three convergent infinite series that should be very easy to sum up at this stage. Provide the value of each sum.

(a) $\sum_{k=0}^{\infty} \frac{(-1)^k (\pi/3)^{2k}}{(2k)!} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

(b) $\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^k = \frac{3/2}{1 - (1/2)} = 3$

(c) $\sum_{k=0}^{\infty} \frac{(\ln(2))^k}{k!} = e^{\ln(2)} = 2$

8. (20 pts.) Here are five easy antiderivatives to evaluate.

(a)

$$\int \sec(5x)\ dx = \frac{1}{5}\ln\left|\tan(5x) + \sec(5x)\right| + C$$

(b)

$$\int \frac{4}{x^2 - x}\ dx = \int \frac{4}{x(x-1)}\ dx = \int \frac{4}{x-1} - \frac{4}{x}\ dx = 4\ln\left|\frac{x-1}{x}\right| + C$$

(c)

$$\int \frac{\sin^3(x)}{\cos(x)}\ dx = \int \frac{\sin(x)(1-\cos^2(x))}{\cos(x)}\ dx$$

$$= \int \tan(x)\ dx - \int \sin(x)\cos(x)\ dx$$

$$= \ln\left|\sec(x)\right| - \frac{1}{2}\sin^2(x) + C$$

(d)

$$\int \frac{2x-1}{x^2+1}\ dx = \int \frac{2x}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx = \ln(x^2+1) - \tan^{-1}(x) + C$$

(e)

$$\int x^2\cos(x)\ dx = x^2\sin(x) - \int 2x\sin(x)\ dx$$

$$= x^2\sin(x) - \left[(2x)\cdot(-\cos(x)) - \int 2(-\cos(x))\ dx\right]$$

$$= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

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9. (10 pts.) Find the area under the curve

$$y = \sqrt{25 - x^2}$$

from x = -5 to x = 2. (a) First write the definite integral whose numerical value is the area. (b) Then evaluate the definite integral. *[WARNING: This is not 1/2 or 1/4 of a disk.]*

$$\mathit{Area} = \int_{-5}^{2} \sqrt{25 - x^2}\ dx = 5\int_{-5}^{2} \sqrt{1 - \left(\tfrac{x}{5}\right)^2}\ dx$$

$$= 25\int_{\alpha}^{\beta} \cos^2(\theta)\ d\theta, \quad \textit{where} \quad \begin{cases} \frac{x}{5} = \sin(\theta), & dx = 5\cos(\theta)\,d\theta, \\ \alpha = \sin^{-1}(-1), & \beta = \sin^{-1}(2/5) \end{cases}$$

$$= \frac{25}{2}\int_{\alpha}^{\beta} 1 + \cos(2\theta)\ d\theta$$

$$= \frac{25}{2}\left(\theta + \sin(\theta)\cos(\theta)\right)\Big|_{\alpha}^{\beta}$$

$$= \frac{25}{2}\left(\beta + \sin(\beta)\cos(\beta) - \alpha - \sin(\alpha)\cos(\alpha)\right)$$

$$= \frac{25}{2}\left(\sin^{-1}\left(\frac{2}{5}\right) + \frac{\pi}{2}\right) + \sqrt{21}$$

10. (10 pts.) (a) (2 pts.) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc.

$$\frac{4x^2-5}{x^3(x-2)(25x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2} + \frac{Ex+F}{25x^2+1} + \frac{Gx+H}{(25x^2+1)^2}$$

(b) (3 pts.) Obtain the numerical values of the literal constants *A*, *B*, and *C* in the partial fraction decomposition given below.

$$(*) \qquad \frac{2x^2+3x-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

A, *B*, and *C* satisfy (*) for each *x* different from zero if, and only if

$$(A+B)x^2 + Cx + 4A = 2x^2 + 3x - 8$$

for every real number *x*. Equating coefficients and solving the resulting linear system results in *A* = -2, *B* = 4, and *C* = 3.//

(c) (5 pts.) If one were to integrate the rational function in part (a), one probably would encounter the integral below. Find this integral.

$$\int \frac{x}{(25x^2+1)^2}\,dx = \frac{1}{50}\int u^{-2}\,du \qquad \text{when } u = 25x^2+1$$

$$= -\frac{1}{50}u^{-1} + C = -\frac{1}{50}(25x^2+1)^{-1} + C$$

11. (10 pts.) Suppose

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\,10^k}(x-5)^k$$

(a) By using sigma notation and term-by-term differentiation *as done in class*, obtain a power series for *f′*(*x*).

$$f'(x) = \frac{d}{dx}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k10^k}(x-5)^k\right] = \sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{(-1)^{k+1}}{k10^k}(x-5)^k\right]$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k10^k}\frac{d}{dx}\left[(x-5)^k\right] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{10^k}(x-5)^{k-1}$$

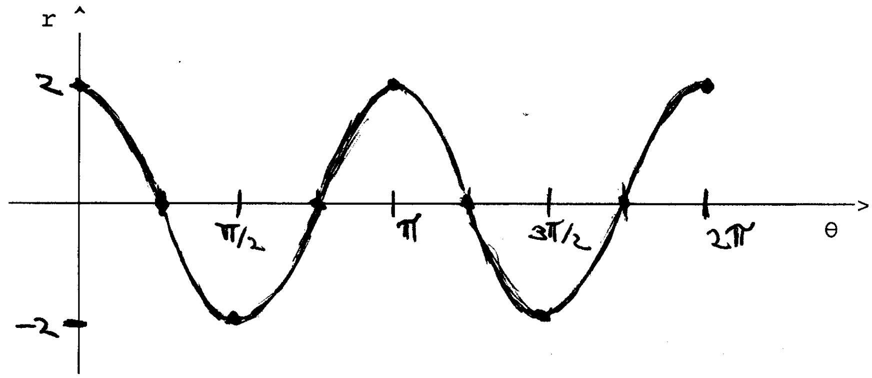
(b) By using sigma notation and integrating term-by-term *as done in class*, obtain an infinite series whose sum has the same numerical value as that of the following definite integral. [We are working with the power series *f* above.]

$$\int_5^7 f(x)\,dx = \int_5^7 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k10^k}(x-5)^k\,dx = \sum_{k=1}^{\infty} \int_5^7 \frac{(-1)^{k+1}}{k10^k}(x-5)^k\,dx$$

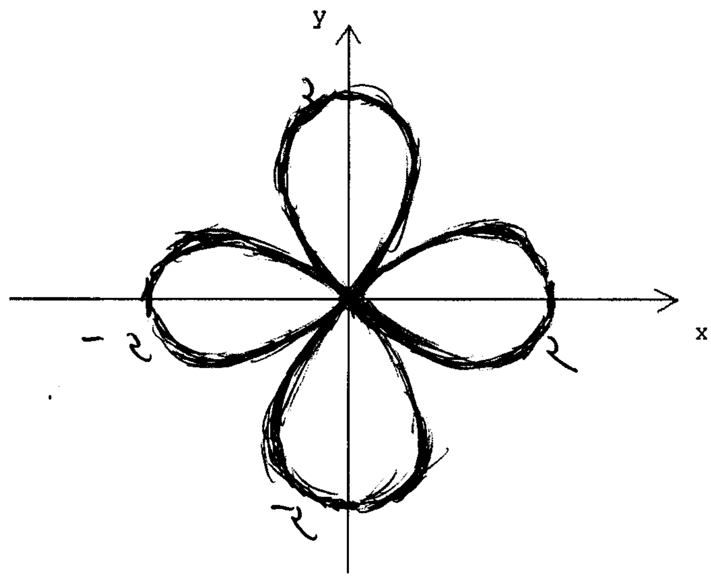
$$= \sum_{k+1}^{\infty} \frac{(-1)^{k+1}}{k10^k}\int_5^7 (x-5)^k\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}2^{k+1}}{k(k+1)10^k}$$

12, (6 pts.) Sketch the curve $r = 2\cos(2\theta)$ in polar coordinates.
Do this as follows: (a) Carefully sketch the auxiliary curve, a rectangular graph, on the r,θ-coordinate system provided. (b) Then translate this graph to the polar one.

(a)



(b) [Think of polar coordinates lying over the x,y axes below.]



(c) Write down, but do not attempt to evaluate a definite integral that provides the numerical value of the area of one-half of one of the rose petals above.

$$Area = \int_0^{\pi/4} \frac{(2\cos(2\theta))^2}{2}\, d\theta$$

13. (4 pts.) Evaluate the following integral:

$$\int_0^{\infty} 2xe^{-x}\,dx = \lim_{b\to\infty}\int_0^{b} 2xe^{-x}\,dx = \lim_{b\to\infty}\left((-2xe^{-x})\Big|_0^b - \int_0^b (-2)e^{-x}\,dx\right)$$

$$= \lim_{b\to\infty}\left(2 - \frac{2}{e^b} - \frac{2b}{e^b}\right) = 2$$

Silly 10 point bonus: You may do exactly one of the following:
(a) State the Mean-Value Theorem for Integrals and use it to prove the second part of the Fundamental Theorem of Calculus, or
(b) Identify the function f given by the power series in Problem 11 above.
State which bonus you are attempting and where your work is.