READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds. Eschew obfuscation.*

1. (5 pts.) Evaluate the following using the 1st Part of the Fundamental Theorem of Calculus.

$$\int_1^9 \frac{1}{x\sqrt{x}}\,dx = (-2x^{-1/2})\Big|_1^9 = \ldots = 4/3$$

2. (5 pts.) Using a complete sentence and appropriate notation, state precisely the First Part of the Fundamental Theorem of Calculus. [This is sometimes called the *Evaluation Theorem*.]

If $f(x)$ is continuous on $[a,b]$ and $g(x)$ is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

3. (5 pts.) Express the following sum using sigma notation, but do not attempt to find its numerical value.

$$-\frac{1}{2}+\frac{1}{4}-\frac{1}{6}+\frac{1}{8}-\frac{1}{10}+\frac{1}{12} = \sum_{k=1}^{6}\frac{(-1)^k}{2k}$$

4. (5 pts.) Obtain the upper limit of summation and rewrite the summand in order to make the following equation true. **Do not attempt to evaluate the sum.**

$$\sum_{k=8}^{45} 2^{3k} = \sum_{j=0}^{37} 2^{3j+24}$$

since $j = k - 8$ is equivalent to $k = j + 8$.

5. (5 pts.) Express the following limit as a definite integral. Do not attempt to evaluate the integral.

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (2\cos^3 x_k^*)\,\Delta x_k \;; \quad a = -\pi,\; b = 2\pi.$$

$$L = \int_{-\pi}^{2\pi} 2\cos^3(x)\,dx$$

6. (5 pts.) Using complete sentences and appropriate notation, state the Second Part of the Fundamental Theorem of Calculus.

// Let f(x) be a function that is continuous on an interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\, dt,$$

for each x in I, then $g'(x) = f(x)$ for each x in I.//

7. (10 pts.) Find each of the following derivatives.

(a) $\frac{d}{dx}\left[\int_x^{\pi} 4\sin(t^3)\, dt \right] = -4\sin(x^3)$

(b) $\frac{d}{dx}\left[\int_{-1}^{x^2} \sqrt{t+1}\, dt \right] = (2x)\sqrt{x^2+1}$

8. (5 pts.) (a) (3 pts.) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: To start, complete the sentence,"ln(x) = ...")

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

for $x > 0$. The domain of the natural log function is $(0,\infty)$, and its range is the whole real line, $(-\infty,\infty)$.

(b) (2 pts.) Given ln(a) = 4 and ln(b) = -3, evaluate the following integral.

$$\int_1^{ab^2} \frac{1}{t}\, dt = \ln(a) + 2\ln(b) = 4 + (2)(-3) = -2$$

9. (5 pts.) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable **t**, so the differential denoting the variable of integration is ***dt***. DO NOT ATTEMPT TO EVALUATE THE DEFINITE INTEGRAL IN *t* THAT YOU OBTAIN.

$$\frac{dy}{dx} = 4e^{x^2}, \quad y(\pi/4) = 1.$$

$$y(x) = 1 + \int_{\pi/4}^{x} 4e^{t^2}\, dt$$

10. (5 pts.) Evaluate the definite integral below by expressing it in terms of u correctly and then evaluating the resulting integral using a formula from geometry.

$$\int_1^{e^4} \frac{\sqrt{16-(\ln(x))^2}}{x}\,dx \;;\; u = \ln(x)$$

$$\int_1^{e^4} \frac{\sqrt{16-(\ln(x))^2}}{x}\,dx = \int_0^4 \sqrt{16-u^2}\,du = 4\pi$$

since $du = 1/x\ dx$. The right-most integral provides the area of one fourth of a circle with a radius of 4 centered at the origin.

11. (10 pts.) Write each of the following two sums in closed form.

(a) $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{2} - \frac{1}{n+2}$

(b) $\displaystyle\sum_{k=0}^{100}\left(\frac{1}{5^k}\right) = \frac{5}{4}\left[1 - \left(\frac{1}{5}\right)^{101}\right]$

12. (10 pts.) If the function *f* is continuous on [a,b], then the *net signed area* A between $y = f(x)$ and the interval [a,b] is defined by

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x.$$

Reveal all the details in computing the numerical value of the net signed area of $f(x) = 4x^3$ over the interval [0,1] using only the definition above with

$$x_k^*$$

the right end point of each subinterval in the regular partition. *Do not use the Fundamental Theorem, Part 1.*

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} 4\left(\frac{k}{n}\right)^3\frac{1}{n}$$
$$= \lim_{n\to\infty}\left(\frac{4}{n^4}\right)\sum_{k=1}^{n} k^3 = \lim_{n\to\infty}\left(\frac{4n^2(n+1)^2}{4n^4}\right)$$
$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1$$

since $\Delta x = 1/n$ and $x_k = k/n$ for $k = 0, 1, \ldots$, n are the end points of the intervals of the general regular partition.

---

13. (10 pts.) A particle moves with a velocity of $v(t) = 2\cos(2t)$ along an s-axis. Find the displacement and total distance traveled over the time interval $[\pi/2, 3\pi/4]$.

$$\text{Displacement} = \int_{\pi/2}^{3\pi/4} v(t)\, dt = \int_{\pi/2}^{3\pi/4} 2\cos(2t)\, dt = \ldots = -1$$

$$\text{Total\_Distance} = \int_{\pi/2}^{3\pi/4} |v(t)|\, dt = \int_{\pi/2}^{3\pi/4} |2\cos(2t)|\, dt = \ldots = 1$$

---

14. (5 pts.) Evaluate the following limit:

$$L = \lim_{x \to 0} (1 + x)^{1/(3x)} = e^{1/3}$$

---

15. (10 pts.) Let the function g be defined by the equation

$$g(x) = \int_0^x e^{t^2}\, dt - 5x$$

for $x \in (-\infty, \infty)$. Then since

$$g'(x) = e^{x^2} - 5 = e^{x^2} - e^{\ln(5)} \quad \text{and} \quad g''(x) = 2xe^{x^2} ,$$

(a) $g(0) = \int_0^0 e^{t^2}\, dt - 5(0) = 0$

(b) $g'(0) = -4$

(c) $g''(0) = 0$

(d) Determine the open intervals where g is increasing or decreasing. Be specific.

From properties of the exponential and natural logarithmic functions, it follows that $g' > 0$ when $x^2 > \ln(5)$, and $g' < 0$ when $x^2 < \ln(5)$. Thus, g is increasing when

$$x < -\sqrt{\ln(5)} \quad or \quad \sqrt{\ln(5)} < x ,$$

and g is decreasing when

$$-\sqrt{\ln(5)} < x < \sqrt{\ln(5)} .$$

(e) Determine the open intervals where g is concave up or concave down. Be specific. [This is easy.]

After examining the second derivative, it is easy to see that g is concave up when $x > 0$, and g is concave down when $x < 0$.

Silly 10 Point Bonus: Reveal the magic in evaluating the following limit:

$$\lim_{x \to 0^-} \frac{1}{x}\int_{-x}^{x} \frac{\sec(t)}{\pi^2 + t^4}\, dt$$

[Say where your work is, for it won't fit here.]

An extended discussion of this problem may be found in the TestTomb: Fall, 2005.