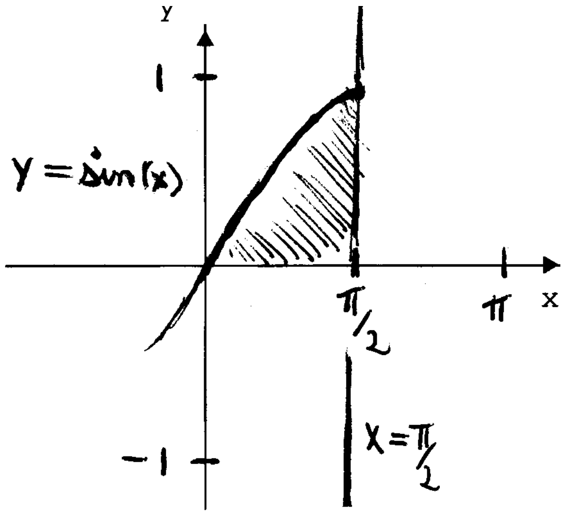
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Eschew obfuscation. Show me all the magic on the page.*

1. (25 pts.) The region R in the first quadrant enclosed by the curves defined by $y = \sin x$, $y = 0$, and $x = \pi/2$ is sketched below for your convenience.



(a) Write down, but do not attempt to evaluate the definite integral whose numerical value gives the area of the region R if one integrates with respect to *x* so the differential in the integral is *dx*.

$$\text{Area} = \int_0^{\pi/2} \sin(x)\ dx$$

(b) Write down, but do not attempt to evaluate the definite integral whose numerical value gives the area of the region R if one integrates with respect to *y* so the differential in the integral is *dy*.

$$\text{Area} = \int_0^1 \frac{\pi}{2} - \sin^{-1}(y)\ dy$$

(c) Using the method of cylindrical shells, write a single definite integral *dy* whose numerical value is the volume of the solid obtained when the region R above is revolved around the x-axis. Do not evaluate the integral.

$$\text{Volume} = \int_0^1 2\pi y\left(\frac{\pi}{2} - \sin^{-1}(y)\right) dy$$

(d) Using the method of disks or washers, write down a definite integral *dx* to compute the same volume as in part (c). Do not evaluate the integrals.

$$\text{Volume} = \int_0^{\pi/2} \pi \sin^2(x)\ dx$$

(e) Write down, but do not attempt to evaluate, the definite integral that gives the arc-length of the curve $y = \ln(x)$ from $x = 1$ to $x = e$.

$$\text{Length} = \int_1^e \sqrt{1 + \left(\frac{1}{x}\right)^2}\ dx$$

or

$$\text{Length} = \int_0^1 \sqrt{1 + e^{2y}}\ dy$$

2. (15 pts.) (a) (5 pts.) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc. Be very careful here.

$$\frac{4x^2+5}{x(3x-5)^3(9x^2+1)^2} = \frac{A}{x} + \frac{B}{3x-5} + \frac{C}{(3x-5)^2} + \frac{D}{(3x-5)^3} + \frac{Fx+G}{9x^2+1} + \frac{Hx+I}{(9x^2+1)^2}$$

(b) (5 pts.) If one were to integrate the rational function in part (a), one might encounter the integral below. Evaluate the integral below.

$$\int \frac{1}{(3x-5)^3}\,dx = \ldots = -\frac{1}{6}(3x-5)^{-2} + C \quad \textit{using } u = 3x-5 \textit{ in a u-substitution.}$$

(c) (5 pts.) If one were to integrate the rational function in part (a), one might also encounter the integral below. Evaluate the integral below.

$$\int \frac{x}{(9x^2+1)^2}\,dx = \ldots = -\frac{1}{18}(9x^2+1)^{-1} + C \quad \textit{using the u-subst. } u = 9x^2+1.$$

3. (60 pts.) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[**6 pts./part**]

(a) $$\int_0^{\sqrt{\pi/3}} 6x\cos(x^2)\,dx = \int_0^{\pi/3} 3\cos(u)\,du = 3\sin(\pi/3) - 3\sin(0) = \frac{3\sqrt{3}}{2}$$

using $u = x^2$.

(b)
$$\int 6x\sin(2x)\,dx = (6x)\left(\frac{-\cos(2x)}{2}\right) - \int (6)\left(\frac{-\cos(2x)}{2}\right)\,dy$$
$$= -3x\cos(2x) + \frac{3}{2}\sin(2x) + C$$

using integration by parts with $u = 6x$ and $dv = \cos(2x)\,dx$.

(c) $$\int 16x^2e^{4x}\,dx = 4x^2e^{4x} - 2xe^{4x} + \frac{1}{2}e^{4x} + C$$

by applying integration by parts twice using $dv = \exp(4x)dx$ each time.

(d) $$\int_0^1 \tan^{-1}(x)\,dx = \left(x\tan^{-1}(x)\right)\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx = \ldots = \frac{\pi}{4} - \ln(\sqrt{2})$$

by integrating by parts using $u = \arctan(x)$ and $dv = dx$.

(e) $$\int \frac{2}{\sqrt{1+4t^2}}\,dt = \ldots = \ln\left|\sqrt{1+4t^2} + 2t\right| + C$$

using the trigonometric substitution $2t = \tan(\theta)$.

(f) $$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \ldots = \frac{\sin^{-1}(x)}{2} - \frac{x\sqrt{1-x^2}}{2} + C$$

using the trigonometric substitution $x = \sin(\theta)$.

3. (Continued) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[**6 pts./part**]
(g)

$$\int \frac{x^4 - 5x^3}{x^2+1}\,dx = \int x^2 - 5x - 1 + \frac{5x+1}{x^2+1}\,dx$$
$$= \frac{1}{3}x^3 - \frac{5}{2}x^2 - x + \frac{5}{2}\ln|x^2+1| + \tan^{-1}(x) + C$$

after doing an elementary long division of polynomials.

(h)

$$\int 4\cos(4t)\cos(2t)\,dt = \int 4\left(\frac{\cos(4t+2t) + \cos(4t-2t)}{2}\right)dt$$
$$= \int 2\cos(6t) + 2\cos(2t)\,dt$$
$$= \frac{1}{3}\sin(6t) + \sin(2t) + C$$

using some obvious trigonometry. This may also be handled by using integration by parts, but that route is much messier.

(i) $$\int \frac{50}{\theta^2 - 25}\,d\theta = \int \frac{5}{\theta-5} - \frac{5}{\theta+5}\,d\theta = 5\ln\left|\frac{\theta-5}{\theta+5}\right| + C$$

after doing an obvious partial fraction decomposition.

(j) $$\int \sec^3(x)\,dx = \frac{1}{2}(\tan(x)\sec(x) + \ln|\tan(x)+\sec(x)|) + C$$

since

$$\int \sec^3(x)\,dx = \int \sec(x)\sec^2(x)\,dx$$
$$= \sec(x)\tan(x) - \int [\sec(x)\tan(x)]\tan(x)\,dx$$
$$= \sec(x)\tan(x) - \int \sec(x)[\sec^2(x) - 1]\,dx$$
$$= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx$$

by using integration by parts and a friendly trigonometry identity.

Silly 10 Point Bonus: (a) Prove that

$$(*) \qquad \frac{1}{t} \geq 1 - \frac{t}{4}$$

for every t ε [1,3]. (b) Using (a) to compare a couple of integrals, prove ln(3) > 1. (c) Using part (b) and the Intermediate-Value Theorem for Continuous Functions, prove that there is a unique number x_0 in the open interval (2,3) where ln(x) = 1. //

(a) By doing elementary algebra, the truth of inequality (*) above on the interval [1,3] may be seen to be equivalent to that of the following inequality:

$$\frac{1}{t} - \left(1 - \frac{t}{4}\right) = \frac{(t-2)^2}{4t} \geq 0$$

for t ε [1,3]. The inequality is actually true when $t > 0$, and false when $t < 0$.

(b) From the order preservation properties of definite integrals, the truth of inequality (*) on the closed interval [1,3] implies that

$$\begin{aligned} \ln(3) = \int_1^3 \frac{1}{t}\,dt &\geq \int_1^3 1 - \frac{t}{4}\,dt \\ &= \left(t - \frac{1}{8}t^2\right)\Big|_1^3 \\ &= (3 - \frac{9}{8}) - (1 - \frac{1}{8}) = 1. \end{aligned}$$

To see that the inequality is actually *sharp*, it suffices to mumble that the two functions

$$f(t) = \frac{1}{t} \quad \text{and} \quad g(t) = 1 - \frac{t}{4}$$

are continuous on [1,3], with $f(t) > g(t)$ except at $x = 2$. Thus the two integrals above are not equal.

(c) Recall that

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

for $x > 0$ and thus is continuous and strictly increasing on the closed interval [2,3]. From (b) above, 1 < ln(3). Since $1/t < 1$ on the interval [1,2], except at t = 1, it follows that ln(2) < 1 from an argument analogous to that in (b) above. From the Intermediate-Value Theorem for Continuous Functions, it follows that there is at least one point x_0 in the open interval (2,3) where ln(x) = 1. Since ln(x) is strictly increasing, it is one-to-one. Thus, there is only one such point.//

Key Questions: What was the point of this exercise, and where did the function g above originate????

Loose End: Prove that if f is continuous on [a,b] with $f(x) \geq 0$ for every x ε [a,b], and there is some x_0 ε [a,b] where f is positive, then

$$\int_a^b f(x)\,dx > 0.$$

[Proof of the 'mumble' may be reduced to dealing with this easily!!]