READ ME FIRST: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page. Eschew obfuscation.

1. (10 pts.) Consider the definite integral below. (a) Write down the sum, S_4, used to approximate the value of the integral below if Simpson's Rule is used with $n = 4$. Do not attempt to evaluate the sum. (b) Write down the sum, T_4, used to approximate the value of the integral below if Trapezoid Rule is used with $n = 4$. Do not attempt to evaluate the sum.

$$\int_2^4 \sqrt{x}\ dx$$

Plainly, $\Delta x = \frac{1}{2}$, *and* $x_k = 2 + \frac{k}{2} = \frac{4+k}{2}$, *for* $k = 0,1,2,3,4.$

are the points of the regular partition we need.

(a) $$S_4 = \frac{1}{3}\Delta x(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) = \frac{1}{6}\left(\sqrt{\tfrac{4}{2}} + 4\sqrt{\tfrac{5}{2}} + 2\sqrt{\tfrac{6}{2}} + 4\sqrt{\tfrac{7}{2}} + \sqrt{\tfrac{8}{2}}\right)$$

(b) $$T_4 = \frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) = \frac{1}{4}\left(\sqrt{\tfrac{4}{2}} + 2\sqrt{\tfrac{5}{2}} + 2\sqrt{\tfrac{6}{2}} + 2\sqrt{\tfrac{7}{2}} + \sqrt{\tfrac{8}{2}}\right)$$

2. (10 pts.) Evaluate the integrals that converge.

(a) $$\int_{-1}^{+\infty} \frac{2\,dx}{x^2+1} = \lim_{b\to\infty}\int_{-1}^{b} \frac{2}{x^2+1}\,dx = \lim_{b\to\infty}\left(2\tan^{-1}(b) - \left(-\frac{\pi}{2}\right)\right) = \frac{3\pi}{2}$$

(b) $$\int_0^{\pi/2} \tan(x)\ dx = \lim_{b\to\pi/2^-}\int_0^{b} \tan(x)\ dx = \lim_{b\to\pi/2^-} \ln(\sec(b)) = +\infty$$

Silly 10 Point Bonus: Prove that for every positive integer $n \geq 1$,

$$\sqrt{n+1} - 1 < \sum_{k=1}^{n} \frac{1}{2\sqrt{k}}$$

Since $f(x) = \frac{1}{2\sqrt{x}}$ is decreasing for $x > 0$, it follows from properties of the definite integral that

$$\int_k^{k+1} \frac{1}{2\sqrt{x}}\ dx < \frac{1}{2\sqrt{k}}$$

for each positive integer $k \geq 1$. Thus, adding these inequalities, we have

$$\sqrt{n+1} - 1 = \int_1^{n+1} \frac{1}{2\sqrt{x}}\ dx = \sum_{k=1}^{n}\int_k^{k+1} \frac{1}{2\sqrt{x}}\ dx < \sum_{k=1}^{n} \frac{1}{2\sqrt{k}}\ .$$

3. (4 pts.) Express the repeating decimal as a fraction, more specifically as a quotient of positive integers. [The fraction does not have to be in lowest terms.]

$$0.54545454\ldots = \frac{54}{99} = \frac{6}{11}$$

either by using the "high school" method or summing an appropriate geometric series.

4. (4 pts.) Find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so, find its limit.

$$0,\ \frac{1}{2^3},\ \frac{2}{3^3},\ \frac{3}{4^3},\ \frac{4}{5^3}\ \ldots.$$

$$a_n = \frac{n-1}{n^3} \quad \text{for } n \geq 1. \quad \text{Thus, } \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n^2} - \frac{1}{n^3}\right) = 0.$$

5. (4 pts.) $$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$$

Use the error estimate from alternating series test to determine a specific value of $n \geq 1$ so that the partial sum s_n approximates $\pi/4$ to 5 decimal places, where, of course,

$$s_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$$

Since

$$\left|s_n - \frac{\pi}{4}\right| < \frac{1}{2n+1} \quad \text{for} \quad n \geq 1,$$

it suffices to find a positive integer n so that

$$\frac{1}{2n+1} \leq \left(\frac{1}{2}\right)10^{-5}$$

is true. Solving this inequality for n and taking into account that n must be a positive integer yields $n \geq 100000$. Thus, take $n = 100000$.

6. (8 pts.) Determine whether the series converges, and if so, find its sum.

(*a*) $$\sum_{k=1}^{\infty}\left(-\frac{2}{3}\right)^{k+2} = \frac{(-2/3)^3}{1-(-2/3)} = -\frac{8}{45}$$

This is obviously a geometric series with $r = -2/3$ and $a = -8/27$. Since $|r| < 1$, the series converges and the computations are easy.

(*b*) $$\sum_{k=1}^{\infty}[\ln(k+2) - \ln(k+3)]$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}[\ln(k+2) - \ln(k+3)] = \lim_{n\to\infty}[\ln(3) - \ln(n+3)] = -\infty$$. So (*b*) diverges.

7. (4 pts.) Use root test to determine whether the series converges. If the test is inconclusive, say so.

$\sum_{k=1}^{\infty}\left(1 + \frac{2}{k}\right)^k$ Since $\rho = \lim_{k\to\infty}\left[\left(1+\frac{2}{k}\right)^k\right]^{1/k} = \lim_{k\to\infty}\left(1+\frac{2}{k}\right) = 1$, root test is inconclusive.

8. (4 pts.) Apply the divergence test and state what it tells you about each of the following series.

(a) $\sum_{k=1}^{\infty}\frac{1}{k!}$ Since $\lim_{k\to\infty}\frac{1}{k!} = 0$, divergence test provides no information concerning the convergence of (a).

(b) $\sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{k}+3}$ Since $\lim_{k\to\infty}\frac{\sqrt{k}}{\sqrt{k}+3} = 1 \neq 0$, divergence test implies that (b) diverges.

9. (4 pts.) Use ratio test to determine whether the series converges. If the test is inconclusive, say so.

$\sum_{k=1}^{\infty}\frac{k}{5^k}$ Since $\rho = \lim_{k\to\infty}\frac{(k+1)/5^{k+1}}{k/5^k} = \lim_{k\to\infty}\frac{k+1}{5k} = \frac{1}{5} < 1$, ratio test implies that the series converges.

10. (4 pts.) Use comparison test to show the following series converges.

$\sum_{k=1}^{\infty}\frac{\sqrt{k}}{k^2+4}$ First,

$$\sqrt{k}/(k^2+4) \le 1/k^{3/2}$$

for $k \ge 1$. Since the p-series $\sum_{k=1}^{\infty}(1/k^{3/2})$ converges, comparison test, implies that the series of problem #10 converges.

11. (4 pts.) Confirm that the integral test is applicable, and then use it to determine whether the following series converges:

$\sum_{k=2}^{\infty}\frac{1}{k\ln^2(k)}$ Let $f(x) = 1/(x\ln^2(x))$ for $x \ge 2$. Plainly f is a positive continuous function, and $1/((k)\ln^2(k)) = f(k)$ for $k \ge 2$. Since $f'(x) = (-1)(x\ln^2(x))^{-2}(\ln^2(x)+2\ln(x)) < 0$ for $x > 2$, it follows that f is decreasing for $x \ge 2$. This means that we may use f, defined above, in integral test to determine whether the given series converges. Since

$$\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln^2(x)}\,dx = \lim_{b\to\infty}\left(\frac{1}{\ln(2)} - \frac{1}{\ln(b)}\right) = \frac{1}{\ln(2)},$$

it follows from integral test that the given series of #11 converges.

12. (4 pts.) Find all values of x for which the series converges, and find the sum of the series for those values of x.

$$\frac{1}{x^2} + \frac{5}{x^3} + \frac{25}{x^4} + \frac{125}{x^5} + \frac{625}{x^6} + \cdots$$

Evidently, this is a geometric series. Writing the series using sigma notation makes things easy. Thus,

$$\sum_{k=0}^{\infty} \frac{5^k}{x^{k+2}} = \sum_{k=0}^{\infty} \left(\frac{1}{x^2}\right)\left(\frac{5}{x}\right)^k = \left(\frac{1}{x^2}\right)\left(\frac{1}{1 - \left(\frac{5}{x}\right)}\right) = \frac{1}{x^2 - 5x}$$

provided that $\left|\frac{5}{x}\right| < 1$, *or* $5 < |x|$, *or* $(x < -5$ *or* $5 < x)$.

13. (6 pts.) Classify each of the following series as absolutely convergent (AC), conditionally convergent (CC), divergent (D), or none of the preceding, (N). Circle the letters corresponding to your choice.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{1/4}+1}$ ~~(AC)~~ **(CC)** ~~(D)~~ ~~(N)~~

(b) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^{3/2}}{k+1}$ ~~(AC)~~ ~~(CC)~~ **(D)** ~~(N)~~

(c) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ **(AC)** ~~(CC)~~ ~~(D)~~ ~~(N)~~.

14. (4 pts.) The following series diverges:

$$(*) \qquad \sum_{k=1}^{\infty} \left[\frac{1}{3k+2} - \frac{1}{k^{3/2}}\right]$$

Provide the indirect reasoning that gives a proof of this fact.
If (*) were convergent, then, since the series

$$(**) \qquad \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

converges, the sum of the series given by (*) and (**) would converge. This means that it would follow that

$$(***) \qquad \sum_{k=1}^{\infty} \frac{1}{3k+2}$$

must converge. That, however, is impossible, since integral test reveals that (***) must diverge, a contradiction. So (*) cannot converge.

15. (6 pts.) [Complete the following.] A p-series is a series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

This series diverges if ___ $p \le 1$ ___ and this series converges if ___ $p > 1$ ___ .

16. (8 pts.) (a) Using complete sentences and appropriate notation, give the precise ε - N definition of

(*) $$\lim_{n \to \infty} a_n = L.$$

// We write (*) above if L is a number such that, for each $\varepsilon > 0$, there is a positive integer N, dependent on ε, such that for every positive integer n, if $n \geq N$, then $|a_n - L| < \varepsilon$. //

(b) Give the precise mathematical definition of the sum of an infinite series,

(**) $$\sum_{k=1}^{\infty} a_k$$

// A number s is the sum of the series (**) above if

$$s = \lim_{n \to \infty} \sum_{k=1}^{n} a_k.$$

If the limit fails to exist, the series is said to diverge. //

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17. (4 pts.) From the definition of a limit of sequence, we know there is a positive integer N so that if $n \geq N$, then

(*) $$\left| \frac{n}{2n+2} - \frac{1}{2} \right| < \left(\frac{1}{2}\right)10^{-3} \quad \text{since} \quad \lim_{n \to \infty} \frac{n}{2n+2} = \frac{1}{2}.$$

Find a positive integer N which works and prove it provides the desired error bound. // By doing a little routine algebra, it is easy to see that when $n \geq 1$, inequality (*) above is equivalent to

$$\left| \frac{-1}{2n+2} \right| < \left(\frac{1}{2}\right)10^{-3} \Leftrightarrow \frac{1}{2n+2} < \left(\frac{1}{2}\right)10^{-3} \Leftrightarrow 1000 < n+1 \Leftrightarrow 999 < n.$$

It follows that if we let $N = 1000$, then if $n \geq N$, then $n > 999$, which is equivalent to (*). Just trace the double-headed arrow path backwards.

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18. (8 pts.) Let the sequence $\{a_n\}$ be defined recursively by

$$a_1 = \sqrt{3}, \quad and \quad a_{n+1} = \sqrt{3 + a_n}$$

for $n \geq 1$. (a) List the first four terms of the sequence.

$$a_1 = \sqrt{3}, \quad a_2 = \sqrt{3+\sqrt{3}}, \quad a_3 = \sqrt{3+\sqrt{3+\sqrt{3}}}, \quad and \quad a_4 = \sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}$$

(b) Assuming the sequence converges, find its limit L.

$$L = \lim_{n \to \infty} a_{n+1} = \sqrt{3 + \lim_{n \to \infty} a_n} = \sqrt{3 + L}$$

implies that $L^2 - L - 3 = 0$, so that $L = (1+\sqrt{13})/2$ or $L = (1-\sqrt{13})/2$. Since the sequence is nonnegative, the limit, if it exists must also be nonnegative. As a consequence, $L = (1+\sqrt{13})/2$.

Silly 10 Point Bonus: Prove that for every positive integer $n \geq 1$,

$$\sqrt{n+1} - 1 < \sum_{k=1}^{n} \frac{1}{2\sqrt{k}}$$

[Say where your work is, for it won't fit here. *Page 1 of 5.*]