
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magical transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds.*

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1. (10 pts.) (a) Write the value of the following limit as a definite integral.

$$L = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \cos\left(\frac{\pi}{2}\left(\frac{k}{n}\right)\right).$$

$$L = \int_0^1 \cos\left(\frac{\pi}{2}x\right)\, dx$$

(b) By evaluating the integral you obtained in part (a) above using the Fundamental Theorem of Calculus, give the exact numerical value of the limit, L.

$$L = \int_0^1 \cos\left(\frac{\pi}{2}x\right)\, dx = \frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\Big|_0^1 = \frac{2}{\pi}.$$

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2. (10 pts.) Evaluate each of the following sums in closed form.

(a) $$\sum_{i=0}^{199}\left(\frac{1}{5}\right)^i = \frac{1-(1/5)^{(199+1)}}{1-(1/5)} = \frac{5}{4}[\,1-(1/5)^{200}\,].$$

(b) $$\sum_{i=1}^{200}(4i-1) = 4\sum_{i=1}^{200} i - \sum_{i=1}^{200} 1 = \frac{4(200)(201)}{2} - 200$$
$$= (200)(401) = 80200.$$

3. (10 pts.) (a) State the Fundamental Theorem of Calculus.

Suppose that f is continuous on the closed interval [a,b].
Part 1: If the function g is defined on [a,b] by

$$g(x) = \int_a^x f(t)\, dt,$$

then g is an antiderivative of f. That is, $g'(x) = f(x)$ for each x in [a,b]. [This, really, is the punch line!!]
Part 2: If G is any antiderivative of f on [a,b], then

$$\int_a^b f(x)\, dx = G(b) - G(a).$$

(b) Write the solution to the following initial value problem in terms of a definite integral with respect to the variable t, but don't attempt to evaluate the definite integral:

$$y'(x) = e^{\tan(x)}, \quad y(\pi/3) = 8.$$

$$y(x) = 8 + \int_{\pi/3}^{x} e^{\tan(t)}\, dt$$

4. (10 pts.) Reveal all the details of evaluating the given integral by computing

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

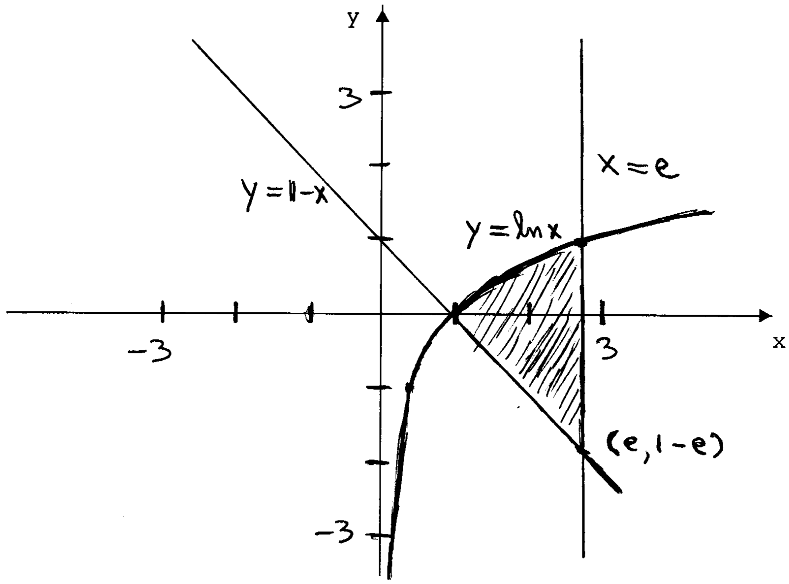
where the sum is assumed to have originated from a regular partition of the given interval of integration. Here, you are to actually compute the Riemann sum in closed form and then evaluate the limit. Suppose that $b > 0$ below. Then

$$\begin{aligned}\int_0^b x^2\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{bi}{n}\right)^2 \frac{b}{n} \\ &= \lim_{n\to\infty} \left(\frac{b^3}{n^3}\right) \sum_{i=1}^{n} i^2 \\ &= \lim_{n\to\infty} \left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \lim_{n\to\infty} \frac{b^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\ &= \frac{b^3}{3}.\end{aligned}$$

since $\Delta x = b/n$ and $x_i = bi/n$ for $i = 0, 1, \ldots, n$ are the end points of the intervals of the general regular partition.

5. (15 pts.) (a) Sketch very carefully the bounded region bounded by the curves y = 1 - x, y = ln x and x = e on the coordinate system provided. Label very carefully. (b) Write down a single integral, dx, that provides the numerical value of the area of the region. (c) Write a sum of definite integrals, dy, that yields the area of the region. Do not attempt to evaluate the definite integrals.

(a)



(b) Area = $\int_1^e \ln(x) - (1 - x)\, dx$

(c) Area = $\int_{1-e}^{0} e - (1 - y)\, dy + \int_0^1 e - e^y\, dy$

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6. (5 pts.) Suppose that g is defined on the interval [π,4π] by means of the equation

$$g(x) = \int_{2\pi}^{x} \frac{\cos(t)}{t}\, dt.$$

Determine the open intervals in (π,4π) where g is increasing or decreasing.

It follows from the Fundamental Theorem of Calculus that g′(x) = cos(x)/x for x in (π,4π). Thus, the sign of g′(x) is determined by the sign of cos(x) on (π,4π). Thus, g′(x) > 0 when 3π/2 < x < 5π/2 or 7π/2 < x < 4π, and g′(x) < 0 when π < x < 3π/2 or 5π/2 < x < 7π/2. It follows that g is increasing on the set (3π/2,5π/2)∪(7π/2,4π) and g is decreasing on the set (π,3/2π)∪(5π/2,7π/2).

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7. (10 pts.) Differentiate the following functions:
[Make sure you label your derivatives correctly.]

(a) $g(x) = \int_0^x \sec^3(t)\,dt + \tan(x)$, *for* $x \in (-\pi/2, \pi/2)$.

$g'(x) = \sec^3(x) + \sec^2(x)$, *for* $x \in (-\pi/2, \pi/2)$.

(b) $f(x) = \int_0^{\ln(x)} \frac{1}{1+t^2}\,dt$, *for* $x > 0$.

$f'(x) = \frac{1}{1+(\ln(x))^2} \cdot \frac{1}{x}$, *for* $x > 0$.

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8. (10 pts.) (a) Using only the second comparison property of integrals, give both a lower bound and an upper bound on the true numerical value of the integral below.

$$I = \int_1^2 \frac{1}{t}\,dt.$$

What does this tell you about ln(2)?

Let $f(t) = t^{-1}$. Then $f'(x) = -t^{-2} < 0$ when t is in the interval (1,2). So f is decreasing on [1,2]. Thus, $1/2 = f(2) \le f(t) \le f(1) = 1$ when $1 \le t \le 2$. From the 2nd comparison property of integrals, it follows that we have $1/2 = (1/2)(1) \le I \le (1)(1) = 1$. Since the numerical value of the integral is ln(2), we see easily that $1/2 \le \ln(2) \le 1$.

(b) Using appropriate properties of the definite integral and suitable area formulas from geometry, evaluate the following definite integral. [Roughly sketching a couple of simple graphs might help. First use linearity, though.]

$$\int_{-2}^{0} x + (4-x^2)^{1/2}dx = \int_{-2}^{0} x\,dx + \int_{-2}^{0} (4-x^2)^{1/2}dx = -2 + \frac{1}{4}\pi(2)^2 = \pi - 2$$

since the numerical value of the first integral is simply the negative of the area of a right triangle with two legs of length 2 and the numerical value of the second integral is one-fourth the area of a circle with radius 2.

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9. (10 pts.) (a) Find the average value of $f(x) = \sec^2(x)$ over the interval $[\pi/4, \pi/3]$.

$$f_{AVE} = \frac{1}{(\pi/3)-(\pi/4)}\int_{\pi/4}^{\pi/3} \sec^2(x)\, dx = \frac{12}{\pi}(\tan(x))\Big|_{\pi/4}^{\pi/3}$$
$$= \frac{12}{\pi}(\tan(\pi/3) - \tan(\pi/4)) = \frac{12}{\pi}[\sqrt{3} - 1].$$

(b) What must the constants *a* and *b* be for the following equation to be valid when we use the substitution $u = \tan(x)$??

$$\int_{-\pi/3}^{\pi/4} \tan^2(x)\sec^2(x)dx = \int_a^b u^2 du.$$

$a = \tan(-\pi/3) = -(3)^{1/2}$ $\quad$ $b = \tan(\pi/4) = 1.$

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10. (10 pts.) Consider the definite integral below. (a) Write down the sum, S_4, used to approximate the value of the integral below if Simpson's Rule is used with $n = 4$.

$$\int_4^6 x^{1/2} dx$$

Since $\Delta x = 1/2$, the endpoints of the regular partition we are using are $x_0 = 8/2$, $x_1 = 9/2$, $x_2 = 10/2$, $x_3 = 11/2$, and $x_4 = 12/2$.

(a) $S_4 = \frac{1}{6}\cdot(4^{1/2} + 4(9/2)^{1/2} + 2(5)^{1/2} + 4(11/2)^{1/2} + 6^{1/2})$

(b) The magnitude of error in using Simpson's Rule to approximate the definite integral of $f(x)$ on an interval $[a,b]$ with *n* subintervals may be estimated using

$$|ES_n| \le \frac{M_4(b-a)^5}{180n^4},$$

provided the fourth derivative of f, $f^{(4)}(x)$, is continuous on $[a,b]$, and $|f^{(4)}(x)| \le M_4$ for each x in $[a,b]$.

Use this information to determine how big *n* must be to ensure that you will obtain an approximation to the true value of the integral above that is accurate to 6 decimal places when using Simpson's Rule.// Since $|f^{(4)}(x)| = (15/16)x^{-7/2}$ is decreasing on $[4,6]$, $|f^{(4)}(x)| \le 15/2^{11}$ on $[4,6]$. Consequently, after a little algebra,

$$|ES_n| \le \frac{1}{3\cdot 2^8\cdot n^4}.$$

Since

$$\frac{1}{3\cdot 2^8\cdot n^4} < \frac{1}{2}10^{-6} \Leftrightarrow \left(\frac{10^6}{3\cdot 2^7}\right)^{1/4} < n,$$

and $2^7 = 128$, $n = 10$ will clearly provide the desired accuracy. [Using a silly calculator, the best you can do is $n = 8$!!]

Silly 10 Point Bonus: Provide a complete proof of the first part of the Fundamental Theorem of Calculus as given by Edwards and Penney. Say where your work is, for it won't fit here.