READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Use complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". The answer really consists of all the magic transformations. Do not "box" your final results. Show me all the magic on the page.*

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1. (5 pts.) Suppose a spring has a natural length of 1 foot and a force of 15 pounds is required to compress the spring to a length of 9 inches. How much work is done in stretching this spring from a its natural length to a length of 18 inches?

Since (1/4)k = 15 implies that k = 60 (lbs./ft.),

$$W = \int_0^{1/2} 60x\,dx = (30x^2)\Big|_0^{1/2} = \frac{30}{4}\ (ft.\text{-}lbs.).$$

Note: A nonstandard solution may be obtained by getting the spring constant in pounds per inch. In this case k = 5. The corresponding integral for work is given by

$$W = \int_0^{6} 5x\,dx = (\frac{5}{2}x^2)\Big|_0^{6} = 90\ (inch.\text{-}lbs.).$$

This is not recommended.

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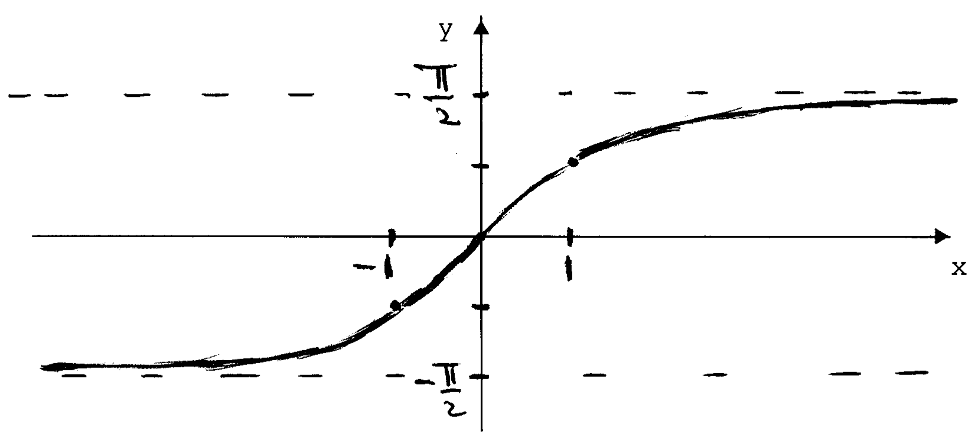
2. (10 pts.)
(a) Sketch the graph of $f(x) = \tan^{-1}(x)$ on the coordinate system provided. Label carefully.



(b) Provide the value of the limit that follows.

$$\lim_{x \to -\infty} \sec^{-1}(x) = \frac{\pi}{2}.$$

Note: If you had done as the instructor instructed and graphed all the inverse trigonometric functions for yourself, then this was a snooze. Otherwise it was hopeless.

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3. (5 pts.) Evaluate the following, quite proper, definite integral:

$$\int_{-1/2}^{1/2} \frac{1}{(1-x^2)^{1/2}}dx = \sin^{-1}(x)\Big|_{-1/2}^{1/2} = \sin^{-1}(1/2)-\sin^{-1}(-1/2) = \frac{\pi}{3}.$$

3. (10 pts.) (a) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc.

$$\frac{x^2+9}{x^2(x-1)(4x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{4x^2+1}.$$

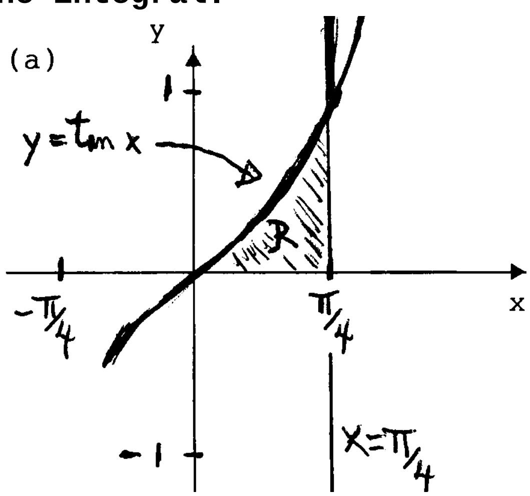
(b) Now obtain the indefinite integral of the rational function of part (a) in terms of the constants A, B, C, etc. using the partial fraction decomposition. Do not attempt to obtain the numerical values of the constants.

$$\int \frac{x^2+9}{x^2(x-1)(4x^2+1)}\,dx = A\ln|x| - Bx^{-1} + C\ln|x-1|$$

$$+\frac{D}{8}\ln|4x^2+1| + \frac{E}{2}\tan^{-1}(2x) + K$$

4. (10 pts.) (a) Sketch the region in the 1st quadrant enclosed by the curves defined by y = tan x, y = 0, and x = π/4. Suppose the region is revolved around the line defined by x = π. (b) Using the method of cylindrical shells, write down the definite integral used to compute the volume of the solid of revolution formed. **Don't evaluate the integral.** (c) Using the method of slicing [disks/washers here], write down the definite integral used to compute the same volume as in part (b). **Don't evaluate the integral.**

(a)



(b)

$$V = 2\pi\int_0^{\pi/4}(\pi - x)\tan(x)\,dx$$

(c)

$$V = \pi\int_0^1 (\pi - \tan^{-1}(y))^2 - (\pi - (\pi/4))^2\,dy$$
$$= \pi\int_0^1 (\pi - \tan^{-1}(y))^2 - \frac{9\pi^2}{16}\,dy$$

5. (10 pts.) Differentiate each of the following functions.

(a) $f(x) = \cos^{-1}(x)$ $\qquad f'(x) = -\dfrac{1}{(1-x^2)^{1/2}}$

(b) $f(x) = \sec^{-1}(x)$ $\qquad f'(x) = \dfrac{1}{|x|(x^2-1)^{1/2}}$

(c) $f(x) = 2^x$ $\qquad f'(x) = \ln(2)\cdot 2^x$

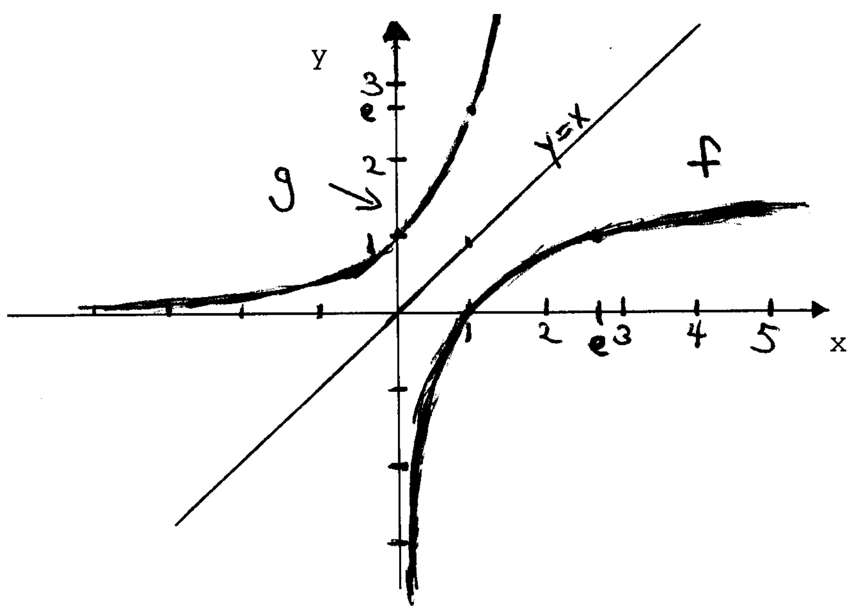
(d) $f(x) = \cot^{-1}(x)$ $\qquad f'(x) = -\dfrac{1}{x^2+1}$

(e) $f(x) = \log_{2\pi}(x)$ $\qquad f'(x) = \dfrac{1}{x\ln(2\pi)}$

6. (10 pts.) (a) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: Complete the sentence,"ln(x) = ...")

$\ln(x) = \int_1^x \frac{1}{t}\,dt$ *for* $x \in (0,\infty)$. The range of the natural logarithm is all of $\mathbb{R} = (-\infty,\infty)$.

(b) Carefully sketch both $f(x) = \ln(x)$ and $g(x) = e^x$ on the coordinate system below. **Label very carefully**.



7. (20 pts.) Here are five trivial trigonometric integrals to evaluate. [4 pts./part]

(a) $$\int \sin^2(x)\,dx = \int \frac{1 - \cos(2x)}{2}\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

(b) $$\int \sec(4x)\,dx = \frac{1}{4}\ln|\sec(4x) + \tan(4x)| + C$$

(c) $$\int \frac{\cos^2(t)}{\sin(t)}\,dt = \int \frac{1 - \sin^2(t)}{\sin(t)}\,dt = \int csc(t) - \sin(t)\,dt$$
$$= -\ln|csc(t) + \cot(t)| + \cos(t) + C.$$

(d) $$\int \sin(4x)\cos(3x)\,dx = \int \frac{\sin(7x) + \sin(x)}{2}$$
$$= -\frac{\cos(7x)}{14} - \frac{\cos(x)}{2} + C.$$

(e) $$\int \tan^4(t)\sec^2(t)\,dt = \frac{\tan^5(t)}{5} + C.$$

Silly 10 Point Bonus:

If $x > 0$, how are $\tan^{-1}(x)$ and $\tan^{-1}(1/x)$ related? Proof??

Let $f(x) = \arctan(x)$ and $g(x) = \arctan(1/x)$ for $x > 0$. Then

$$f'(x) = \frac{1}{1+x^2}$$

and

$$g'(x) = \left[\frac{1}{1 + \left(\frac{1}{x}\right)^2}\right]\cdot\left[\frac{-1}{x^2}\right]$$
$$= \frac{-1}{1 + x^2}$$

for $x > 0$. Consequently, for $x > 0$, $f'(x) + g'(x) = 0$. It follows that there is some constant C such that $f(x) + g(x) = C$ for each $x > 0$. Since $f(1) + g(1) = 2\arctan(1) = \frac{\pi}{2}$, $C = \pi/2$. Hence, $\tan^{-1}(1/x) = \pi/2 - \tan^{-1}(x)$ for each $x > 0$.//

8. (20 pts.) Evaluate each of the following antiderivatives [5 pts./part]

(a)

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx$$
$$= x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + C$$

by integrating by parts using u = $\tan^{-1}(x)$ and dv = $1\cdot dx$ and then doing the little u-substitution that follows in one's biocomputer.

(b)

$$\int \frac{1}{x^2+2x+10}\,dx = \int \frac{1}{(x+1)^2+9}\,dx = \frac{1}{9}\int \frac{1}{1+\left(\frac{x+1}{3}\right)^2}\,dx$$
$$= \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C$$

after doing the obvious u-substitution.

(c)

$$\int (1-9t^2)^{1/2}\,dt = \int (1-(3t)^2)^{1/2}\,dt = \int \cos(\theta)\cdot\frac{1}{3}\cos(\theta)\,d\theta$$
$$= \frac{1}{3}\int \cos^2(\theta)\,d\theta = \frac{1}{3}\int \frac{1+\cos(2\theta)}{2}\,d\theta$$
$$= \frac{1}{6}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$
$$= \frac{1}{6}\left(\sin^{-1}(3t) + 3t(1-9t^2)^{1/2}\right) + C$$

after following the $3t = \sin(\theta)$ trigonometric substitution path.

(d)

$$\int \frac{x^6}{x^3+x}\,dx = \int x^3 - x + \frac{x^2}{x^3+x}\,dx$$
$$= \int x^3 - x + \frac{x}{x^2+1}\,dx$$
$$= \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln(x^2+1) + C$$

after a diversionary long division.

Silly 10 Point Bonus:

If $x > 0$, how are $\tan^{-1}(x)$ and $\tan^{-1}(1/x)$ related? Proof??

Note: An answer may found at the bottom of Page 4 of 5.