
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your final results. Communicate. Show me all the magic on the page.*

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1. (10 pts.) Find Taylor's formula for the given function f at a = π/2. Find both the Taylor polynomial, $P_3(x)$, and the Lagrange form of the remainder term, $R_3(x)$, for the function f(x) = cos(x) at a = π/2. Then write cos(x) in terms of $P_3(x)$ and $R_3(x)$.

$$P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(\pi/2)}{k!}(x-\frac{\pi}{2})^k = -\sin(\pi/2)(x-\frac{\pi}{2}) + \frac{\sin(\pi/2)}{3!}(x-\frac{\pi}{2})^3$$
$$= -(x-\frac{\pi}{2}) + \frac{1}{6}(x-\frac{\pi}{2})^3.$$

$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x-\frac{\pi}{2})^4 = \frac{\cos(z)}{24}(x-\frac{\pi}{2})^4$$

for some z between x and $\frac{\pi}{2}$.

$$\cos(x) = -(x-\frac{\pi}{2}) + \frac{1}{6}(x-\frac{\pi}{2})^3 + \frac{\cos(z)}{24}(x-\frac{\pi}{2})^4$$

for some z between x and $\frac{\pi}{2}$.

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2. (10 pts.) Evaluate each of the following integrals. Look before you leap, for propriety may be problematical.

(a)

$$\int_{\ln(2)}^{\infty} e^{-2x}\,dx = \lim_{t\to\infty}\int_{\ln(2)}^{t} e^{-2x}\,dx$$
$$= \lim_{t\to\infty}\left(-\frac{1}{2}e^{-2x}\right)\Big|_{\ln(2)}^{t}$$
$$= \lim_{t\to\infty}\left(\frac{1}{2}e^{-2\ln(2)} - \frac{1}{2}e^{-2t}\right)$$
$$= \frac{1}{2}e^{-2\ln(2)} = \frac{1}{8}.$$

(b)

$$\int_{0}^{10} (10-x)^{-1/2}\,dx = \lim_{t\to 10^-}\int_{0}^{t}\frac{1}{(10-x)^{1/2}}dx$$
$$= \lim_{t\to 10^-} -2(10-x)^{1/2}\Big|_{0}^{t}$$
$$= \lim_{t\to 10^-}(2(10)^{1/2} - 2(10-t)^{1/2}) = 2(10)^{1/2}.$$

3. (10 pts.) Find the radius of convergence and the interval of convergence of the power series function

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{5^k (k+1)^2} .$$

First observe that the series is centered at x = 2. Then we apply the ratio test for absolute convergence in order to determine the radius of convergence of the power series.

$$\rho(x) = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \ldots = \lim_{k\to\infty} \frac{1}{5}\left(\frac{k+1}{k+2}\right)^2 |x-2| = \frac{1}{5}|x-2|.$$

Now, $\rho(x) < 1$ if, and only if $|x - 2| < 5$. Thus, R = 5 is the radius of convergence. The endpoints are $x_L = -5 + 2$ and $x_R = 2 + 5$. When you substitute x_L into the power series and simplify the algebra, you obtain $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2}$ which converges. When you substitute x_R into the power series and simplify the algebra, you obtain $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2}$ which converges. [The convergence is actually absolute.] I = [-3, 7].

4. (10 pts.) Determine whether the sequence $\{a_n\}$ converges, and find its limit if it does.

(a) $a_n = n \cdot \tan(\frac{4\pi}{n})$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 4\pi \frac{\tan(\frac{4\pi}{n})}{\frac{4\pi}{n}}$$

$$= \lim_{n\to\infty} 4\pi \frac{\sin(\frac{4\pi}{n})}{\frac{4\pi}{n}\cos(\frac{4\pi}{n})} = 4\pi.$$

[You could use L’Hopital’s rule but it is really not needed.]

(b) $a_n = \sum_{k=1}^{n} \frac{4}{k^2+k} = \sum_{k=1}^{n} \left(\frac{4}{k} - \frac{4}{k+1}\right) = \frac{4}{1} - \frac{4}{n+1}.$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(4 - \frac{4}{n+1} \right) = 4.$$

5. (10 pts.) (a) Find a positive integer N such that the

partial sum $$\sum_{n=1}^{N}(-1)^{n+1}\left(\frac{5}{n^3}\right)$$

approximates the sum of the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{5}{n^3}\right)$$

to 5 decimal places, and prove your N actually does what you claim.

Let S denote the sum of the series and S_N denote the Nth partial sum above. Then the error estimate from the alternating series test implies that $|S - S_N| < 5/(N+1)^3$. Consequently, to obtain 5 decimal place accuracy, it suffices to find a positive integer N so that $5/(N+1)^3 \le (1/2)10^{-5}$. Now this last inequality is equivalent to $10^6 \le (N+1)^3$, which in turn, is equivalent to $99 \le N$. [Cube roots preserve order, Folks!] N = 99 does the job.//

(b) Since the third and fourth Maclaurin polynomials for $\sin(x)$ are the same, it follows from Taylor's Theorem that if x is a real number different from 0, we may write

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{\cos(c)}{120}x^5,$$

where c is some number between x and 0. Using this equation, obtain an open interval that is centered at 0 where $\sin(x)$ may be approximated to 5 decimal place accuracy using the polynomial

$$P_4(x) = x - (1/6)x^3.$$

First, from the equation above, it follows that for each real number x different from zero that

$$|\sin(x) - P_4(x)| = |\sin(x) - (x - (1/6)x^3)| = \left|\frac{\cos(c)}{120}x^5\right| \le \frac{|x|^5}{120}.$$

There is no error when x = 0. Thus, to obtain the desired accuracy, then, it suffices to have

$$\frac{|x|^5}{120} < \frac{1}{2}10^{-5}.$$

Now

$$\frac{|x|^5}{120} < \frac{1}{2}10^{-5} \Leftrightarrow |x| < \frac{(60)^{1/5}}{10}.$$

An interval that does the job is $I = (-(60)^{1/5}10^{-1},(60)^{1/5}10^{-1})$.

Note: Even without a calculator it is very easy to see that the interval above contains the interval J = (-0.2, 0.2). How??

6. (5 pts.) With proof, determine whether the given series is conditionally convergent, absolutely convergent, or divergent.

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{1/4}}$$

Since the series of absolute values is given by

$$\sum_{k=1}^{\infty}\frac{1}{k^{1/4}},$$

a divergent p-series, the original series is not absolutely convergent. The series is alternating, however, and plainly

$$k^{-1/4} > (k+1)^{-1/4} \textit{ for each } k \geq 1,$$

$$\textit{and}$$

$$\lim_{k\to\infty} k^{-1/4} = 0.$$

Thus, the alternating series test implies that the original series converges. Consequently, the original series is conditionally convergent.

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7. (5 pts.) Using only the integral test, determine whether the series below converges. Explicitly define the function f(x) used, and verify all the hypotheses of the theorem are true.

$$\sum_{n=1}^{\infty}\frac{2n}{n^2+1}$$

Let $f(x) = 2x/(x^2 + 1)$ for $x \geq 1$. Then f is a positive, continuous function. Since $f'(x) = (2 - 2x^2)/(x^2+1)^2$, $f'(x) < 0$ for $x > 1$. Thus, f is decreasing for $x \geq 1$. Since

$$\int_1^{\infty}\frac{2x}{x^2+1}dx = \lim_{b\to\infty}\int_1^{b}\frac{2x}{x^2+1}dx = \lim_{b\to\infty}[\ln(b^2+1)-\ln(2)] = \infty,$$

integral test implies that the given series diverges.

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10 Point Bonus: Suppose that $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are two positive-termed series with

$$(*) \quad \lim_{n\to\infty}(b_n/a_n) = 0.$$

Does the convergence of one of the series tell you anything about the convergence of the other? With proof, explain how and why. [Hint: Can you tolerate epsilon ennui??]//

Epsilon ennui indeed!! Set $\varepsilon = 1$. The definition of the limit of a sequence when applied to (*) with this epsilon implies that there is a positive integer N such that if $n \geq N$, then $b_n/a_n = |b_n/a_n| < 1$. Consequently $b_n < a_n$ when $n \geq N$.

If the series $\sum_{n=1}^{\infty}a_n$ converges, then the "tail series" $\sum_{n=N}^{\infty}a_n$ converges. Comparison test now implies that $\sum_{n=N}^{\infty}b_n$, a "tail series" of $\sum_{n=1}^{\infty}b_n$, converges. Consequently, $\sum_{n=1}^{\infty}b_n$ converges.

8. (10 pts.) (a) Find the rational number represented by the following repeating decimal.

0.3636363636 ... = 36/99 = 12/33 = 4/11.

This may be obtained by the "High School" method or by summing

the infinite series $\sum_{k=1}^{\infty} \frac{36}{(10^2)^k} = \ldots = \frac{36}{100}\left[\frac{1}{1-\left(\frac{1}{100}\right)}\right] = \ldots .$

(b) Find all values of x for which the given geometric series converges, and then express the closed form sum of the series as a function of x.

$$\sum_{k=0}^{\infty} \frac{(x-2)^{k+1}}{10^k} = \sum_{k=0}^{\infty} \frac{(x-2)}{1} \cdot \left(\frac{x-2}{10}\right)^k = \frac{10(x-2)}{12-x}$$

provided $|(x - 2)/10| < 1$ or $|x - 2| < 10$. As an interval what you have in hand is I = (-8, 12).

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9. (10 pts.) Using either comparison test or limit comparison test, determine whether each of the following series converges.

(a) $\sum_{n=1}^{\infty} \frac{3n^2+5}{4n+n^3}$

Since

$$\frac{1}{n} \le \frac{3n^2+5}{4n+n^3} \Leftrightarrow \frac{4n+n^3}{n} \le 3n^2+5 \Leftrightarrow 4+n^2 \le 3n^2+5$$

for $n \ge 1$, and $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, is divergent,

comparison test implies that the series of (a) diverges.

(b) $\sum_{n=1}^{\infty} \frac{24 \cdot \sin^2(n)}{n^2+n}$

Since

$$\frac{24\sin^2(n)}{n^2+n} \le \frac{24}{n^2}$$

for $n \ge 1$, and $\sum_{n=1}^{\infty} \frac{24}{n^2}$ is a positive multiple of a convergent

p-series and thus convergent, comparison test implies that the series in (b) converges.

You may also use the limit comparison test here to deal with either or both of (a) and (b). For (b), though, the use of the limit comparison is more subtle. To deal with the limit requires the use of the squeezing or pinching theorem for sequences and the "end-point case" treated in the bonus problem.

10. (15 pts.) (a) Using known power series, obtain a power series representation for the function f(x) = cos(x^2). Write your answer using sigma notation.

$$f(x) = \cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!}$$

simply by substituting x^2 into the Maclaurin series for cosine.

(b) Beginning with the power series function

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k x^k$$

defined for x satisfying $|x| < 4$, differentiate termwise to find the series representation for f′(x). Write your answer using sigma notation.

$$f'(x) = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k k x^{k-1} = \sum_{j=0}^{\infty} (j+1)\left(\frac{1}{4}\right)^{j+1} x^j$$

by performing the usual term-wise differentiation dance. Can you give me an alias for f′ and tell me where it lives??

(c) Find a power series representation for the function f(x) below by doing termwise integration. Write your answer using sigma notation.

$$f(x) = \int_0^x \frac{1}{1 - t^2}\, dt = \int_0^x \sum_{k=0}^{\infty} (t^2)^k\, dt$$
$$= \int_0^x \sum_{k=0}^{\infty} t^{2k}\, dt = \sum_{k=0}^{\infty} \int_0^x t^{2k}\, dt = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}.$$

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11. (5 pts.) Using divergence test, show that the series

$$\sum_{n=1}^{\infty} \frac{n^3}{2n^2+1}$$

diverges.

Since $\lim_{n\to\infty} \frac{n^3}{2n^2+1} = \lim_{n\to\infty} \frac{n}{2+\frac{1}{n^2}} = \infty \neq 0$, divergence test implies the series above diverges.

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10 Point Bonus: Suppose that

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

are two positive-termed series with

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

Does the convergence of one of the series tell you anything about the convergence of the other? With proof, explain how and why. Say where your work is, for it won't fit here. [Hint: Can you tolerate epsilon ennui??] Look on Page 4 of 6.