NAME:

STUDENT NUMBER: **EXAM NUMBER:**

READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (25 pts.) (a) Using a complete sentence, state the first part of the Fundamental Theorem of Calculus.

(b) Using complete sentences, state the second part of the Fundamental Theorem of Calculus.

(c) Compute $g'(x)$ when $g(x)$ is defined by the following equation.

$$g(x) = \int_2^x e^{4t^2}\,dt + \sec(x)\tan(x)$$

$g'(x) =$

(d) Write the following in terms of a definite integral with respect to the variable t: an antiderivative, h(x), of the function $f(x) = e^{\sin(x)}$ with $h(\pi/4) = 0$. What is the natural domain of h(x) ??

h(x) =

(e) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable **t**, so the differential denoting the variable of integration is **dt**. Do not attempt to evaluate the definite integral you get.

$$\frac{dy}{dx} = \sec^5(x) \text{ with } y(0) = 2\pi$$

$y(x) =$

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2. (18 pts.) Each of the following power series functions is the Maclaurin series of some well-known function. In each case, (i) identify the function, and (ii) provide the interval in which the series actually converges to the function.

(a) $\displaystyle\sum_{k=0}^{\infty} x^k =$

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} =$

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} =$

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} =$

(e) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} =$

(f) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} =$

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3. (7 pts.) Suppose that

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)8^k}(x-3)^k$$

Find the radius of convergence and the interval of convergence of the power series function f.

4. (15 pts.) Suppose that

$$g(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k 10^k}(x-5)^k$$

for all x in (-5, 15).

(a) By using sigma notation and doing term-by-term differentiation, obtain a power series for $g'(x)$. **Write your answer using sigma notation.**

$g'(x) =$

(b) To sketch the graph of the function g near a = 5, you would need g(5), $g'(5)$, and $g''(5)$. All of these may be read from the power series defining g. Provide these values below and then make a sketch of g near a = 5.

$g(5) =$ $g'(5) =$ $g''(5) =$

(c) Integrate term-by-term to obtain an infinite series whose sum is the same as the numerical value of the following definite integral. [We are working with the g of part (a). **Use sigma notation to write your answer** .]

$$\int_5^7 g(x)\, dx =$$

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5. (10 pts.) (a) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc.

$$\frac{2x^2+4}{x(x+1)^3(x^2+1)^2} =$$

(b) If one were to integrate the rational function in part (a), one probably would encounter the integral below. Reveal, in detail, how to evaluate this integral.

$$\int \frac{1}{(x^2+1)^2}\, dx =$$

6. (50 pts.) Obtain the exact numerical value of each of the following if possible. If, by chance, a limit fails to exist or an improper integral or an infinite series fails to converge, say so. Watch your step. Some definite integrals are improper. Sequential limits could involve Riemann sums -- or not. (5 pts./part)

(a) $\int_0^{\pi/6} 8\theta\cos(\theta)\, d\theta =$

(b) $\int_0^{(\pi/6)^{1/2}} 8\theta\cos(\theta^2)\, d\theta =$

(c) $\int_0^1 \frac{2x^2}{(1-x^2)^{1/2}}\, dx =$

(d) $\sum_{k=1}^{\infty} 10\left(-\frac{4}{5}\right)^k =$

(e) $\int_{\pi/2}^{2\pi} |\cos(x)|\, dx =$

6. Obtain the exact numerical value of each of the following if possible...

(f) $\lim_{n \to \infty} \frac{\ln(1+n^5)}{\ln(n)} =$

(g) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sec\left(\frac{\pi}{6}\left(\frac{k}{n}\right)\right) =$

(h) $\lim_{n \to \infty} \sum_{k=6}^{n} \frac{18}{k^2 - k} =$

(i) $\int_6^\infty \frac{18}{x^2 - x}\, dx =$

(j) $\int_1^2 \sec^{-1}(x)\, dx =$

7. (42 pts.) Here are six antiderivatives to evaluate. (7 pts./part)

(a) $\displaystyle\int \frac{\cos^2(x)}{\sin(x)}\,dx =$

(b) $\displaystyle\int x^2\ln(x)\,dx =$

(c) $\displaystyle\int \sin^3(x)\,dx =$

(d) $\displaystyle\int \frac{4}{x^3+x}\,dx =$

(e) $\displaystyle\int x^2e^x\,dx =$

(f) $\displaystyle\int \frac{4x^4}{x^2+1}\,dx =$

8. (6 pts.) Obtain the second Taylor polynomial $p_2(x)$ of the function

$$f(x) = \tan^{-1}(x)$$

at $x_0 = 1$.

9. (6 pts.) Show how to find an interval that is symmetric about the origin where $\sin(x)$ can be approximated by

$$p(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

with four decimal place accuracy.

10. (6 pts.) It turns out that

$$\int_0^1 \sin(x^2)\,dx = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{(2k+1)!}\cdot\frac{1}{4k+3}\right]$$

To approximate the numerical value of the integral above to 4 decimal places by hand, what finite sum

$$S_n = \sum_{k=0}^{n}\left[\frac{(-1)^k}{(2k+1)!}\cdot\frac{1}{4k+3}\right]$$

should you use? Proof??

11. (6 pts.) (a) By doing appropriate algebra and substituting into an appropriate Maclaurin series, obtain the Maclaurin series for the function

$$f(x) = \frac{1}{8 - x^3}$$

(b) What is the interval of convergence for the Maclaurin series of f??

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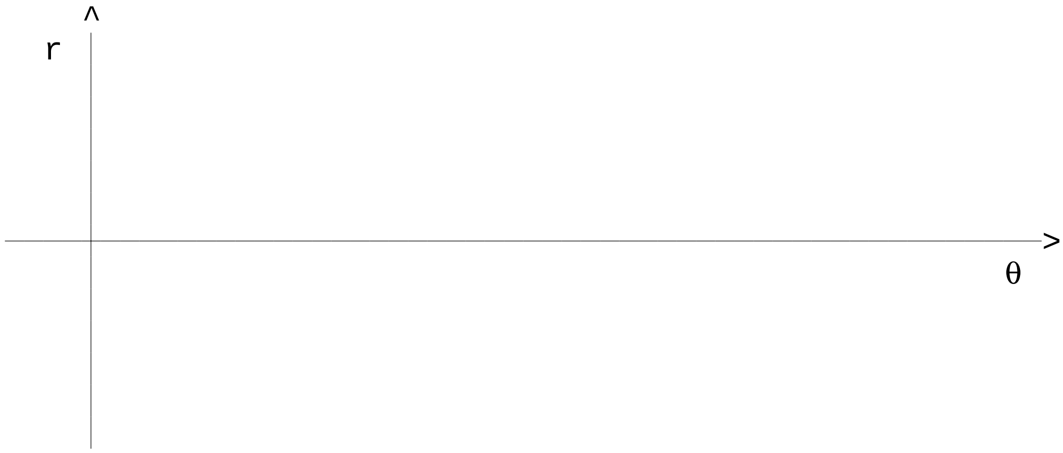
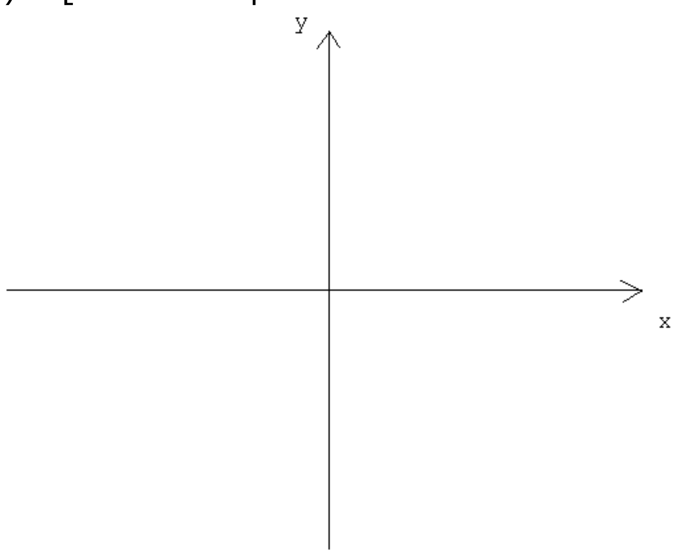
12 (9 pts.) Sketch the curve $r = 2 \cdot \sin(2\theta)$ in polar coordinates, and then compute the area of the region enclosed by one loop of the curve. Do this as follows: (a) Carefully sketch the auxiliary curve, a rectangular graph, on the r,θ-coordinate system provided. (b) Then translate this graph to the polar one. Finally, (c) setup the required polar integral and evaluate it.

(a)



(b) [Think of polar coordinates overlaying the x,y axes below.]



(c)

Area =

Silly 20 point bonus: Let $f(x) = \ln((1 + x)/(1 - x))$. It is easy to show

$$\ln\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^{\infty} \frac{2x^{2k+1}}{2k+1}$$

for $|x| < 1$ using a known Maclaurin series, and that $\ln(2) = f(1/3)$. Using the triangle inequality for absolute values, when $|x| < 1$, it follows that

$$\left| \ln\left(\frac{1+x}{1-x}\right) - \sum_{k=0}^{n} \frac{2x^{2k+1}}{2k+1} \right| \le \sum_{k=n+1}^{\infty} \frac{2|x|^{2k+1}}{2k+1}.$$

By dominating the right side by a geometric series, obtain a useful estimate of the error that depends on x. Then determine how big n needs to be to get an approximation for ln(2) that is accurate to 3 decimal places.