READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds. Eschew obfuscation.*

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1. (5 pts.) Express the following sum using sigma notation, but do not attempt to find its numerical value.

$$(2^3+3^4)-(3^3+3^5)+(4^3+3^6)-(5^3+3^7)+(6^3+3^8)-(7^3+3^9) = \sum_{k=2}^{7}(-1)^k(k^3+3^{k+2})$$

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2. (5 pts.) Obtain the upper limit of summation and rewrite the summand in order to make the following equation true. **Do not attempt to evaluate the sum.** // Since $j = k - 8$,

$$\sum_{k=9}^{55} 10^{2k} = \sum_{j=1}^{47} 10^{2(j+8)}.$$

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3. (5 pts.) If the function f is continuous on [a,b], then the *net signed area* A between $y = f(x)$ and the interval [a,b] is defined by

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Reveal all the details in obtaining the numerical value of the net signed area of $f(x) = 2x$ over the interval [0,4] using the definition above with

$$x_k^*$$

the right end point of each subinterval in the regular partition.

Set $\Delta x = 4/n$. We build a regular partition with points given by $x_k = 0 + k\Delta x$ for $k = 0, 1, 2, ..., n$. The n subintervals created by the partition are given by $[x_{k-1}, x_k]$ for $k = 1, 2, ..., n$. Since

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} 2(k\Delta x)\Delta x = 2(\Delta x)^2\left[\sum_{k=1}^{n} k\right] = 2\left(\frac{16}{n^2}\right)\left(\frac{n(n+1)}{2}\right) = 16\left(1+\frac{1}{n}\right),$$

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty} 16\left(1+\frac{1}{n}\right) = 16.$$

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4. (5 pts.) Express the following limit as a definite integral. Do not attempt to evaluate the integral.

$$L = \lim_{\max \Delta x_k \to 0}\sum_{k=1}^{n} 4x_k^*(1-3x_k^*)\Delta x_k \ ; \quad a = -3,\ b = 4.$$

$$L = \int_{-3}^{4} 4x(1-3x)\,dx.$$

5. (10 pts.) Suppose that f(x) is an integrable function such that

$$\int_{-2}^{1} f(x)\,dx = 4 \quad and \quad \int_{1}^{2} f(x)\,dx = -2.$$

Use this information and appropriate formulas from geometry to evaluate the following definite integrals:

(a) $\int_{1}^{2} 6f(x)\,dx = 6\int_{1}^{2} f(x)\,dx = (6)(-2) = -12.$

(b) $\int_{-2}^{1} 4 - 2f(x)\,dx = \int_{-2}^{1} 4\,dx - 2\int_{-2}^{1} f(x)\,dx = (4)(3) - (2)(4) = 4.$

(c) $\int_{-2}^{2} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx = 4 + (-2) = 2.$

(d) $\int_{-2}^{2} f(x) + (4 - x^2)^{1/2}\,dx = \int_{-2}^{2} f(x)\,dx + \int_{-2}^{2} (4 - x^2)^{1/2}\,dx = 2 + 2\pi.$

(d) $\int_{2}^{1} f(x)\,dx = -\int_{1}^{2} f(x)\,dx = 2.$

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6. (5 pts.) Using a complete sentence and appropriate notation, state precisely the First Part of the Fundamental Theorem of Calculus. [This is sometimes called the *Evaluation Theorem*.]

If f(x) is continuous on [a,b] and g(x) is any antiderivative of f on [a,b], then

$$\int_{a}^{b} f(x)\,dx = g(b) - g(a).$$

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7. (5 pts.) Find the area under the curve

$$y = \frac{1}{(x+4)^2}$$

over the interval [-2,1].

Area $= \int_{-2}^{1} \frac{1}{(x+4)^2}\,dx = -\frac{1}{x+4}\Big|_{-2}^{1} = (-\frac{1}{5})-(-\frac{1}{2}) = \frac{3}{10}.$

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8. (5 pts.) If the given integral is expressed as an equivalent integral in terms of the variable u using the substitution $u = x^3 - 4$ so that the equation below is true, what are the numerical values of the new limits of integration α and β, and what is the new integrand, $f(u)$??? Obtain these but do not attempt to evaluate the du integral.

$$\int_{-1}^{2} \pi x^2(x^3-4)^5\, dx = \int_{\alpha}^{\beta} f(u)\, du$$

$\alpha = (-1)^3 - 4 = -5$

$\beta = (2)^3 - 4 = 4$

$f(u) = (\pi/3)u^5$ since $3x^2dx = du$ implies $x^2dx = (1/3)du$.

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9. (5 pts.) Using complete sentences and appropriate notation, state the Second Part of the Fundamental Theorem of Calculus.
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Let $f(x)$ be a function that is continuous on an interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_{a}^{x} f(t)\, dt,$$

for each x in I, then $g'(x) = f(x)$ for each x in I.

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10. (10 pts.) Let the function g be defined by the equation

$$g(x) = \int_{3^{1/2}}^{x} \tan^{-1}(t)\, dt$$

for $x \in (-\infty, \infty)$. Then

(a) $g(\sqrt{3}) = \int_{3^{1/2}}^{3^{1/2}} \tan^{-1}(t)\, dt = 0.$

(b) $g'(\sqrt{3}) = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$ since $g'(x) = \tan^{-1}(x)$.

(c) $g''(\sqrt{3}) = \dfrac{1}{4}$ since $g''(x) = \dfrac{1}{1+x^2}$.

(d) Determine the open intervals where g is increasing or decreasing. Be specific.

From the details of the answer to (b) above, g is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. [Key Question: Where is $\tan^{-1}$ positive or negative ??]

(d) Determine the open intervals where g is concave up or concave down. Be specific.

From the details of the answer to (c) above, g is concave up on $(-\infty,\infty)$, i.e., always.

11. (5 pts.) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable **t**, so the differential denoting the variable of integration is **dt**. Do not attempt to evaluate the definite integral you get.

$$\frac{dy}{dx} = \sec^3(x) \text{ with } y(-\pi/4) = 2$$

$$y(x) = 2 + \int_{-\pi/4}^{x} \sec^3(t)\,dt.$$

12. (10 pts.) A particle moves with a velocity of $v(t) = \cos(t)$ along an s-axis. Find the displacement and total distance traveled over the time interval $[\pi/2, 2\pi]$.

$$\text{Displacement} = \int_{\pi/2}^{2\pi} v(t)\,dt = \int_{\pi/2}^{2\pi} \cos(t)\,dt$$
$$= \sin(2\pi) - \sin(\pi/2) = -1.$$

$$\text{Total\_Distance} = \int_{\pi/2}^{2\pi} |v(t)|\,dt = \int_{\pi/2}^{2\pi} |\cos(t)|\,dt$$
$$= \int_{\pi/2}^{3\pi/2} |\cos(t)|\,dt + \int_{3\pi/2}^{2\pi} |\cos(t)|\,dt$$
$$= \int_{\pi/2}^{3\pi/2} -\cos(t)\,dt + \int_{3\pi/2}^{2\pi} \cos(t)\,dt$$
$$= \ldots = 3.$$

13. (5 pts.) (a) (3 pts.) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: Complete the sentence,"ln(x) = ...")

$$\ln(x) = \int_{1}^{x} \frac{1}{t}\,dt$$

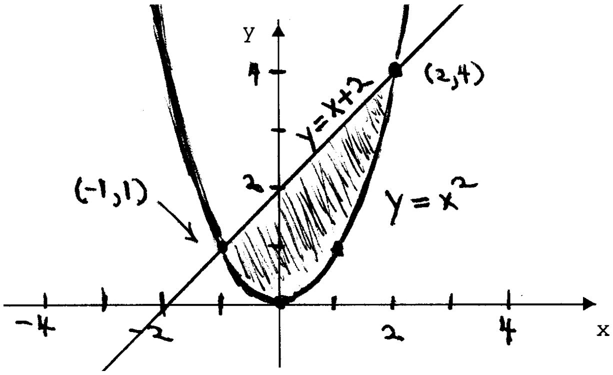
for x > 0. The domain of the natural log function is $(0,\infty)$, and its range is the whole real line, $(-\infty,\infty)$.

(b) (2 pts.) Given ln(a) = 4 and ln(c) = 5, evaluate the following integral.

$$\int_{c}^{a^4} \frac{1}{\omega}\,d\omega = \ln(a^4) - \ln(c) = 4\ln(a) - \ln(c) = (4)(4) - 5 = 11.$$

14. (10 pts.) (a) Sketch the region enclosed by the curves defined by $y = x^2$ and $y = x + 2$. (b) Then give a single definite integral dx whose numerical value is the area of the region. (c) Finally, evaluate the integral of part (b).

(a)



Obviously, to obtain the points of intersection, it is best to solve the system of equations

$$\begin{cases} y = x^2 \\ \text{and} \\ y = x+2. \end{cases}$$

Doing so is elementary algebra.

(*b*) $\text{Area} = \int_{-1}^{2} (x + 2) - x^2\, dx$

(*c*) $= \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Big|_{-1}^{2}$

$= \ldots = \frac{9}{2}.$

15. (5 pts.) Express the function g(x) below in a piecewise defined form that does not involve an integral.

$$g(x) = \int_0^x |t|\, dt = \begin{cases} \dfrac{x^2}{2}, & \text{for } x \geq 0\,; \\ -\dfrac{x^2}{2}, & \text{for } x < 0 \end{cases}$$

since $x \geq 0$ implies $g(x) = \int_0^x |t|\, dt = \int_0^x t\, dt = \ldots = \frac{x^2}{2}$,

and $x < 0$ implies $g(x) = \int_0^x |t|\, dt = \int_0^x -t\, dt = \ldots = -\frac{x^2}{2}$.

16. (5 pts.) Evaluate the following limit:

$$L = \lim_{x\to\infty}\left(1 + \frac{3}{x}\right)^{2x} = \lim_{u\to\infty}\left(1 + \frac{1}{u}\right)^{6u} = e^6 \text{ using } \frac{1}{u} = \frac{3}{x}.$$

Silly 10 Point Bonus: (a) Prove that the function F(x) defined by the equation

$$F(x) = \int_0^x \frac{1}{1+t^2}\, dt + \int_0^{1/x} \frac{1}{1+t^2}\, dt$$

is constant on the interval $(0,\infty)$. (b) Using (a), prove that for $x > 1$ that

$$\int_1^x \frac{1}{1+t^2}\, dt < \int_0^1 \frac{1}{1+t^2}\, dt.$$