READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds. Eschew obfuscation.*

Silly 10 Point Bonus: (a) Prove that the function F(x) defined by the equation

$$F(x) = \int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt$$

is constant on the interval (0,∞). (b) Using (a), prove that for x > 1 that

$$\int_1^x \frac{1}{1+t^2}\,dt < \int_0^1 \frac{1}{1+t^2}\,dt.$$

(a) Plainly the function F defined above is differentiable on the whole real line, (-∞,∞), and it follows from the Fundamental Theorem of Calculus that

$$\begin{aligned} F'(x) &= \frac{1}{1+x^2} + \left[\frac{1}{1+\left(\frac{1}{x}\right)^2}\right]\left[\frac{-1}{x^2}\right] \\ &= \frac{1}{1+x^2} + \frac{-1}{x^2+1} \\ &= 0. \end{aligned}$$

Theorem 5.1.2, a consequence of the Mean-Value Theorem, implies that F is constant on (-∞,∞).

(b) It turns out that to obtain Part (b), it is useful to reveal the identity of the constant. After looking at the structure of F, plainly

$$F(1) = \int_0^1 \frac{1}{1+t^2}\,dt + \int_0^{1/1} \frac{1}{1+t^2}\,dt = 2\int_0^1 \frac{1}{1+t^2}\,dt.$$

What this means is that

(1) $$\int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt = 2\int_0^1 \frac{1}{1+t^2}\,dt$$

is true for each $x > 0$.

To obtain the inequality of Part (b), observe that (1) above is equivalent to

$$\int_0^1 \frac{1}{1+t^1}\,dt + \int_1^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt = 2\int_0^1 \frac{1}{1+t^2}\,dt$$

which, in turn, is equivalent to

(2) $$\int_1^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt = \int_0^1 \frac{1}{1+t^2}\,dt.$$

When $x > 1$,

(3) $$\frac{x}{1+x^2} \le \int_0^{1/x} \frac{1}{1+t^2}\,dt \le \frac{1}{x}$$

since $g(x) = (1 + x^2)^{-1}$ is decreasing for x > 0. Consequently, since the left side of (3) is positive, (b) follows.

Lagniappe:

The set of equations (1) and (2) when combined with the string of inequalities (3) above allows us to see easily that

$$\lim_{x \to \infty} \int_0^x \frac{1}{1+t^2}\,dt = 2\int_0^1 \frac{1}{1+t^2}\,dt$$

and

$$\lim_{x \to \infty} \int_1^x \frac{1}{1+t^2}\,dt = \int_0^1 \frac{1}{1+t^2}\,dt.$$

The nice thing about this is that the string of inequalities in (3) above provides us with an nice estimate of the rate at which this convergence occurs. Of course you do realize that this tells us something about the graph of a certain trigonometric function that we won’t mention by name.

Here now is small problem to puzzle over: Can you prove the following integrals have the same numerical value without invoking the despised trigonometric varmints??

$$A = \int_0^1 \frac{1}{1+t^2}\,dt$$

and

$$B = \int_0^1 (1 - t^2)^{1/2}\,dt \quad ?$$