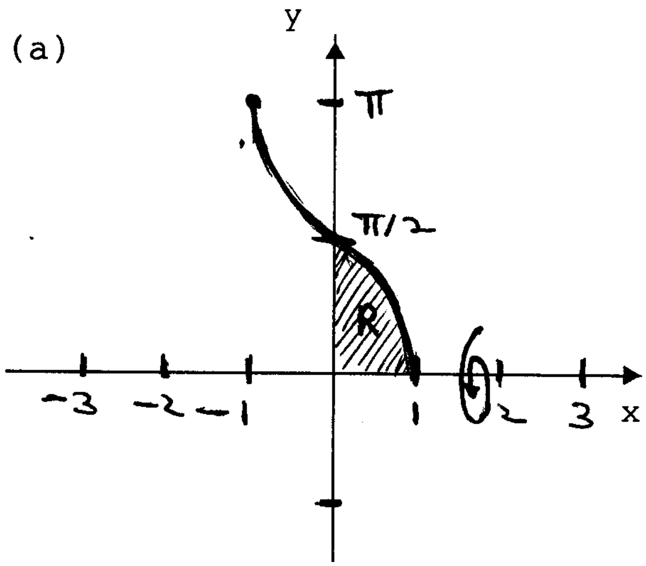
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (10 pts.) (a) Sketch the region in the first quadrant enclosed by the curves $y = \cos^{-1}(x)$, $x = 0$, and $y = 0$. (b) Using the method of disks or washers, write a single definite integral whose numerical value is the volume of the solid obtained when the region is revolved around the x-axis. **Do not evaluate the integral.** (c) Using the method of cylindrical shells, write down a definite integral to compute the same volume as in part (b). **Do not evaluate the integral.**

(a)



(b) For $-1 \le x \le 1$ and $0 \le y \le \pi$

$y = \cos^{-1}(x) \iff x = \cos(y)$.

$$V = \int_0^1 \pi(\cos^{-1}(x))^2\,dx$$

(c)

$$V = \int_0^{\pi/2} 2\pi y\cos(y)\,dy$$

2. (10 pts.) (a) Write down, but do not attempt to evaluate the definite integral that gives the arc length along the curve defined by the equation $y = (1/2)x^2$ from $x = -(3)^{1/2}$ to $x = (3)^{1/2}$.

$$L = \int_{-3^{1/2}}^{3^{1/2}} (1 + x^2)^{1/2}\,dx$$

(b) Suppose a spring has a natural length of 1 foot, and a force of 10 pounds is needed to compress the spring to a length of 8 inches. Write down the definite integral that gives the work done in stretching this spring from its natural length to a total length of 15 inches but do not attempt to evaluate the integral.

$\text{Work} = \int_0^{1/4} 30x\,dx$ since $(1/3)k = 10$ implies $k = 30$ using the standard Hooke's model.

3. (10 pts.) Consider the definite integral below. (a) Write down the sum, S_4, used to approximate the value of the integral below if Simpson's Rule is used with n = 4. **Do not attempt to evaluate the sum. Be very careful.** (b) Write down the sum, T_4, used to approximate the value of the integral below if Trapezoid Rule is used with n = 4. **Do not attempt to evaluate the sum. Be very careful.**

$$\int_{20}^{21} x^{1/4}\, dx$$

Plainly, $\Delta x = \frac{1}{4}$, *and* $x_k = 20 + \frac{k}{4} = \frac{80+k}{4}$, *for* $k = 0,1,2,3,4$

are the points of the regular partition we need.

(a)

$$S_4 = \frac{1}{3}\Delta x(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$
$$= \frac{1}{12}\left((\tfrac{80}{4})^{1/4} + 4(\tfrac{81}{4})^{1/4} + 2(\tfrac{82}{4})^{1/4} + 4(\tfrac{83}{4})^{1/4} + (\tfrac{84}{4})^{1/4} \right)$$

(b)

$$T_4 = \frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$$
$$= \frac{1}{8}\left((\tfrac{80}{4})^{1/4} + 2(\tfrac{81}{4})^{1/4} + 2(\tfrac{82}{4})^{1/4} + 2(\tfrac{83}{4})^{1/4} + (\tfrac{84}{4})^{1/4} \right)$$

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4. (10 pts.) (a) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc.

$$\frac{2x^2+4}{x(x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} + \frac{Gx+H}{(x^2+1)^3}$$

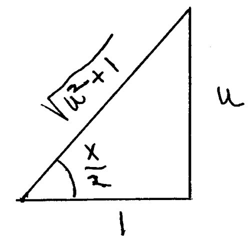
(b) If one were to integrate the rational function in part (a), one probably would encounter the integral below. Reveal, in detail, how to evaluate this integral.

$$\int \frac{x}{(x^2+1)^3}\, dx = \frac{1}{2}\int u^{-3}\, du = -\frac{1}{4}u^{-2} + C = -\frac{1}{4(x^2+1)^2} + C$$

by using the obvious substitution $u = x^2 + 1$. If you are feeling feisty, you can also handle this with a trigonometric substitution.

5. (10 pts.) (a) In making the substitution u = tan(x/2), show all the details in obtaining sin(x) and cos(x) in terms of u by correctly completing the equations below. [**Hint: It helps to draw a right triangle to represent the equation u = tan(x/2).**] (b) Show how to obtain the differential dx in terms of du by solving for x in the equation u = tan(x/2) and differentiating. (c) Using a sentence or two, explain what this substitution does to integrals of rational functions of sine and cosine.

(a) (4 pts.)



$\sin(x/2) = u/(u^2 + 1)^{1/2}$

$\cos(x/2) = 1/(u^2 + 1)^{1/2}$

$\sin(x) = \sin(2(x/2)) = 2u/(u^2 + 1)$

$\cos(x) = \cos(2(x/2)) = (1 - u^2)/(u^2 + 1)$

(b) (4 pts.)

u = tan(x/2) implies $x = 2\tan^{-1}(u)$.

So $dx = (2/(u^2 + 1))du$.

(c) (2 pt.) This substitution transforms integrals of rational functions of sine and cosine into integrals of rational functions of u.

6. (10 pts.) Here are five trivial trigonometric integrals to evaluate. [2 pts./part]

(a) $\int \tan(x)\,dx = \ln|\sec(x)| + C$

(b) $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

(c) $\int \sin^2(x)\,dx = \int \frac{1 - \cos(2x)}{2}\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$

(d) $\int csc(x)\,dx = -\ln|csc(x) + \cot(x)| + C$

(e) $\int \cot(x)\,dx = -\ln|csc(x)| + C$

7. (40 pts.) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.
[**5 pts./part**]

(a)

$$\int_0^{(\pi/4)^{1/2}} (8x)\sin(x^2)\,dx = \int_0^{\pi/4} 4\sin(u)\,du$$
$$= -4\cos(u)\Big|_0^{\pi/4}$$
$$= -4\cos(\pi/4) + 4\cos(0) = 4 - 2\sqrt{2}$$

using the u-substitution $u = x^2$.

(b)

$$\int 2x\tan^{-1}(x)dx = x^2\tan^{-1}(x) - \int\frac{x^2}{1+x^2}dx$$
$$= x^2\tan^{-1}(x) - \int 1 - \frac{1}{x^2+1}\,dx$$
$$= x^2\tan^{-1}(x) - x + \tan^{-1}(x) + C$$

using parts with $u = \tan^{-1}(x)$ and $dv = 2x\,dx$, followed by a long division.

(c)

$$\int\frac{\sin^2(t)}{\cos(t)}dt = \int\frac{1 - \cos^2(t)}{\cos(t)}dt$$
$$= \int\sec(t) - \cos(t)\,dt$$
$$= \ln|\sec(t) + \tan(t)| - \sin(t) + C$$

(d)

$$\int (8x)\sin(x)\,dx = (8x)(-\cos(x)) - \int(8)(-\cos(x))dx$$
$$= -8x\cos(x) + 8\sin(x) + C$$

using integration by parts with $u = 8x$ and $dv = \sin(x)dx$.

7. (Continued) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals.

(e)

$$\begin{aligned}\int x^2e^x dx &= x^2e^x - \int 2xe^x dx\\ &= x^2e^x - (\ 2xe^x - \int 2e^x dx\ )\\ &= x^2e^x - 2xe^x + 2e^x + C\end{aligned}$$

by integrating by parts twice. Keep picking on the exponential varmint as your recognized derivative.

(f)

$$\begin{aligned}\int \frac{1}{(x^2+1)^{1/2}}\, dx &= \int \frac{\sec^2(\theta)}{\sec(\theta)}\, d\theta\\ &= \int \sec(\theta)\, d\theta\\ &= \ln|\sec(\theta) + \tan(\theta)| + C\\ &= \ln|x + (x^2+1)^{1/2}| + C\end{aligned}$$

using the trigonometric substitution $x = \tan(\theta)$.

(g)

$$\begin{aligned}\int_0^1 \ln(x^2+1)\, dx &= x\ln(x^2+1)\big|_0^1 - \int_0^1 x\left(\frac{2x}{x^2+1}\right)dx\\ &= \ln(2) - \int_0^1 2 - \frac{2}{x^2+1}\, dx\\ &= \ln(2) - 2 + \frac{\pi}{2}\end{aligned}$$

using integration by parts with $u = \ln(x^2 + 1)$ and $dv = 1dx$ followed by long division.

(h)

$$\begin{aligned}\int (1-t^2)^{1/2}\, dt &= \int \cos^2(\theta)d\theta\\ &= \int \frac{1+\cos(2\theta)}{2}\, d\theta\\ &= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C\\ &= \frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2} + C\\ &= \frac{1}{2}(\ \sin^{-1}(t) + t(1-t^2)^{1/2}\ ) + C\end{aligned}$$

using the obvious trigonometric substitution $t = \sin(\theta)$. Of course you could also try integration by parts, but htat route is a little thornier.

Silly 10 Point Bonus: The magnitude of error in using Simpson's Rule to approximate the definite integral of $f(x)$ on an interval [a,b] with n subintervals may be estimated using

$$\left|\ \int_a^b f(x)\,dx - S_n\ \right| = |ES_n| \le \frac{K_4(b-a)^5}{180n^4},$$

provided the fourth derivative of f, $f^{(4)}(x)$, is continuous on $[a,b]$, and $|f^{(4)}(x)| \le K_4$ for each x in [a,b]. If Simpson's rule is used to approximate the definite integral giving the exact value of ln(2), reveal how to determine how big must n be to ensure you have 2 decimal place accuracy. [Say where your work is for there isn't room here. Remember n must be even.]