READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page. Eschew obfuscation.*

1. (20 pts.) Obtain the exact numerical value of each of the following if possible. If a limit doesn't exist or an improper integral or an infinite series fails to converge, say so.

(a)

$$\int_2^\infty \frac{10}{x(x+1)}\,dx = \lim_{b\to\infty}\int_2^b \frac{10}{x}-\frac{10}{x+1}\,dx = \lim_{b\to\infty} 10\ln\left(\tfrac{x}{x+1}\right)\Big|_2^b$$

$$= \lim_{b\to\infty}\left[\,10\ln\left(\tfrac{b}{b+1}\right) - 10\ln\left(\tfrac{2}{3}\right)\,\right] = 10\ln\left(\tfrac{3}{2}\right).$$

(b)

$$\sum_{k=2}^\infty \frac{10}{k(k+1)} = \lim_{n\to\infty}\sum_{k=2}^n\left[\,\frac{10}{k} - \frac{10}{k+1}\,\right] = \lim_{n\to\infty}\left[\,\frac{10}{2} - \frac{10}{n+1}\,\right] = 5.$$

(c)

$$\int_1^2 \frac{1}{2-x}\,dx = \lim_{b\to 2^-}\int_1^b \frac{1}{2-x}\,dx = \lim_{b\to 2^-}\int_1^{2-b}\frac{-1}{u}\,du$$

$$= \lim_{b\to 2^-}\int_{2-b}^1 \frac{1}{u}\,du = \lim_{b\to 2^-}[\ \ln(1) - \ln(2-b)\ ] = \infty$$

by using the substitution $u = 2 - x$.

(d)

$$\sum_{k=0}^\infty 10\left(-\frac{3}{4}\right)^k = \frac{10}{1-\left(-\frac{3}{4}\right)} = \frac{40}{7}.$$

(e)

$$\lim_{n\to\infty}\left(1 + \frac{\ln(3)}{n}\right)^n = e^{\ln(3)} = 3.$$

To see this, go show that

$$\lim_{x\to\infty}\left(1 + \frac{A}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+Ax^{-1})} = e^A$$

by using the continuity of the exponential function and showing

$$\lim_{x\to\infty} x\ln(1+Ax^{-1}) = \lim_{x\to\infty}\frac{\ln(1+Ax^{-1})}{x^{-1}} = A.$$

2. (20 pts.) With proof, determine whether each of the following infinite series converge. If a series converges, do not attempt to obtain its sum.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{10}{2+3k^2}$ Since

$$\frac{10}{2+3k^2} \le \frac{10}{3k^2} \text{ for } k \ge 1, \text{ and } \sum_{k=1}^{\infty} \frac{10}{3k^2}$$

is a nonzero multiple of a convergent p-series, and so convergent, comparison test implies that the series in (a) converges.//

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{10}{2+(3/k^2)}$ Since

$$\lim_{k\to\infty} \frac{10}{2+(3/k^2)} = 5 \neq 0,$$

divergence test implies that the series in (b) diverges.//

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{3k!}$ Since

$$\rho = \lim_{k\to\infty} \frac{10^{k+1}}{3(k+1)!}\cdot\frac{3k!}{10^k} = \lim_{k\to\infty} \frac{10}{k+1} = 0,$$

ratio test implies that the series in (c) converges.//

(d) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{2+(3/k^2)}\right)^k$ Since

$$\rho = \lim_{k\to\infty} \left[\left(\frac{1}{2+(3/k^2)}\right)^k\right]^{1/k} = \lim_{k\to\infty} \frac{1}{2+(3/k^2)} = \frac{1}{2} < 1,$$

root test implies that the series in (d) converges.//

(e) $\displaystyle\sum_{k=2}^{\infty} \frac{10}{k\ln^2(k)}$ First, set

$$f(x) = \frac{10}{x\ln^2(x)} \text{ for } x \in [2,\infty).$$

Plainly, f is positive and continuous on the interval $[2,\infty)$. Since

$$f'(x) = \frac{-10(\ln^2(x) + 2\ln(x))}{(x\ln^2(x))^2} < 0 \text{ for } x \in (2,\infty),$$

f is decreasing on $[2,\infty)$. We may use integral test now. Since

$$\int_2^{\infty} \frac{10}{x\ln^2(x)}dx = \lim_{b\to\infty}\int_2^{b} \frac{10}{x\ln^2(x)}dx = \lim_{b\to\infty}\left[\frac{10}{\ln(2)} - \frac{10}{\ln(b)}\right] = \frac{10}{\ln(2)},$$

integral test implies that the series in (e) converges.//

3. (12 pts.) Each of the following power series functions is the Maclaurin series of some well-known function. In each case, (i) identify the function, and (ii) provide the interval in which the series actually converges to the function.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin(x)$ for $x \in (-\infty,\infty)$.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = \tan^{-1}(x)$ for $x \in [-1,1]$.

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \cos(x)$ for $x \in \mathbb{R} = (-\infty,\infty)$.

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = \ln(1+x)$ for $x \in (-1,1]$.

(e) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x = \exp(x)$ for $x \in \mathbb{R} = (-\infty,\infty)$.

(f) $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $x \in (-1,1)$.

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4. (8 pts.) Suppose that

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 7^k}(x-1)^k$$

Find the radius of convergence and the interval of convergence of the power series function f.// To use ratio test for absolute convergence, we compute

$$\rho(x) = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2 \frac{1}{7}|x-1| = \frac{1}{7}|x-1|.$$

Plainly,

$$\rho(x) < 1 \iff \tfrac{1}{7}|x-1| < 1 \iff |x-1| < 7.$$

Thus, the radius of convergence is R = 7. By unwrapping the rightmost inequality above, we can obtain the interior of the interval of convergence, namely, the interval (-6,8). Substitution of x = -6 into f yields

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$$

which converges. Also substitution of x = 8 into f yields

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

which also converges. The interval of convergence: I = [-6,8].

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5. (5 pts.) Obtain the second Taylor polynomial $p_2(x)$ of the function

$$f(x) = x^{1/3}$$

at $x_0 = 8$.

Plainly,

$$f'(x) = (1/3)x^{-2/3} \text{ and } f''(x) - -(2/9)x^{-5/3}$$

for $x \neq 0$. Thus,

$$f^{(0)}(8) = 2,\ f^{(1)}(8) = \tfrac{1}{12},\ \text{ and } f^{(2)}(8) - -(2/9)(8)^{-5/3} = -\tfrac{1}{144}.$$

Consequently, the second Taylor polynomial at $x_0 = 8$ is

$$p_3(x) = \sum_{k=0}^{2} \frac{f^{(k)}(8)}{k!}(x-8)^k = 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2.$$

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6. (5 pts.) **Using sigma notation and an appropriate Maclaurin series**, by doing term-by-term integration, obtain an infinite series that is equal to the numerical value of the following definite integral.

$$\int_0^1 \sin(x^2)\,dx = \int_0^1 \left[\sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!}\right]dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\int_0^1 x^{4k+2}\,dx$$

$$= \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{(2k+1)!}\cdot\frac{1}{4k+3}\right]$$

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7. (5 pts.) Suppose

$$f(x) = \sum_{k=1}^{\infty} \frac{2\pi(x-4)^k}{k20^k}$$

for every $x \in (-16,24)$. By differentiating f term-by-term, obtain a power series function that is the same as $f'(x)$. **Use sigma notation.**

$$f'(x) = \sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{2\pi(x-4)^k}{k20^k}\right] = \sum_{k=1}^{\infty} \frac{2\pi k(x-4)^{k-1}}{k20^k} = \sum_{k=1}^{\infty} \frac{2\pi(x-4)^{k-1}}{20^k}.$$

This is actually a tame geometric varmint that you can easily put in closed form.

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8. (5 pts.) Express .112112... (repeating) as the ratio of two positive integers. [**The ratio does not have to be in lowest terms.**]

.112112... = 112/999 in a couple of ways.

9. (5 pts.)

$$(*) \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{3/4}}$$

Prove the infinite series above is conditionally convergent.

Since

$$\sum_{k=1}^{\infty} |(-1)^{k+1}k^{-3/4}| = \sum_{k=1}^{\infty} \frac{1}{k^{3/4}},$$

a divergent p-series, the series (*) above is not absolutely convergent. Obviously (*) is an alternating series. Observe that

$$\lim_{k\to\infty} \frac{1}{k^{3/4}} = 0, \quad \text{and} \quad \frac{1}{k^{3/4}} > \frac{1}{(k+1)^{3/4}}$$

for $k \geq 1$. Thus, alternating series test implies that (*) converges. Since (*) converges, but not absolutely, (*) is conditionally convergent.

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10. (5 pts.) It turns out that

$$\int_0^1 e^{-x^2}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}.$$

To approximate the numerical value of the integral above to 2 decimal places by hand, what finite sum

$$s_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)k!}.$$

should you use? Proof??

It's not horribly difficult to see the infinite series above actually satisfies the hypotheses of alternating series test. Consequently, we may use the error bound from the alternating series test to determine a partial sum of the series that provides the desired accuracy. Since we then have

$$\left| \int_0^1 e^{-x^2}dx - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)k!} \right| < \frac{1}{(2n+3)(n+1)!}$$

for $n \geq 0$, it suffices to obtain an integer $n \geq 0$ with

$$\frac{1}{(2n+3)(n+1)!} \leq \tfrac{1}{2}10^{-2}.$$

This last inequality is equivalent to

$$200 \leq (2n+3)\cdot(n+1)!$$

By building a small table of factorials, it is easy to see immediately that $n + 1 = 6$ or $n = 5$ is large enough just by using only the factorial term. With a little more tinkering where we take into account the other factor as well, we can see that $n = 3$ does the job. In this case, we actually have $(2n+3)(n+1)! = (9)(24) > 200$.//

11. (5 pts.) (a) By substitution into an appropriate Maclaurin series, obtain the Maclaurin series for the function

$$f(x) = \tan^{-1}(2x^2)$$

(b) What is the domain of the function f?
(c) What is the interval of convergence for the Maclaurin series of f??

(a) We can obtain the Maclaurin series of f by substituting $2x^2$ into the series for $\tan^{-1}(x)$ thus:

$$\begin{aligned} f(x) &= \tan^{-1}(2x^2) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(2x^2)^{2k+1}}{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+2}}{2k+1} \; . \end{aligned}$$

We have convergence of the series to f provided that $2x^2 \in [-1,1]$, or equivalently $|2x^2| \le 1$.
(b) Since the domain of $\tan^{-1}$ is the whole real line, $\mathbb{R} = (-\infty,\infty)$, it follows that the domain of f is also $\mathbb{R} = (-\infty,\infty)$.
(c) It turns out that the set of numbers for which $|2x^2| \le 1$ is true is the same as that for which

$$|x| \le \tfrac{1}{2^{1/2}}.$$

The interval of convergence is $I = [-2^{1/2}/2,\ 2^{1/2}/2]$.//

12. (5 pts.) Show how to find an interval that is symmetric about the origin where $\cos(x)$ can be approximated by $p(x) = 1 - x^2/2$ with two decimal place accuracy.

Having diligently done our homework, we recognize that $p(x)$ above is the third Maclaurin polynomial of cosine. Since the fourth derivative of $\cos(x)$ is $\cos(x)$, and $|\cos(x)| \le 1$ for all x, we may use the Remainder Estimation Theorem to deduce that

$$|\cos(x) - p(x)| = |\cos(x) - p_3(x)| \le \frac{|x|^4}{4!}$$

for every real number x. Consequently, to obtain the desired accuracy, it suffices to ensure that

$$\frac{|x|^4}{4!} < \tfrac{1}{2}10^{-2}.$$

Evidently, this inequality true precisely when

$$|x| < ((1.2)10^{-1})^{1/4}.$$

This means that an appropriate interval is $I = (-(.12)^{1/4},(.12)^{1/4})$. It should be noted that this interval contains $J = (-0.12,0.12)$ which might be more palatable for calculation on a desert island.//

Silly 10 Point Bonus: Obtain an infinite series that gives the exact value of $\tan^{-1}(2)$. Say where your work is, for it won't fit here!!