READ ME FIRST: Show me all the magic very neatly on the page, for I do not read minds. Use correct notation when presenting your computations and arguments. Use complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Eschew obfuscation. Each problem is worth 10 points.

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1. Using a complete sentence and appropriate notation, state precisely the First Part of the Fundamental Theorem of Calculus. [This is sometimes called the *Evaluation Theorem*.]

If $f(x)$ is continuous on $[a,b]$ and $g(x)$ is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

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2. Evaluate the following using the 1st Part of the Fundamental Theorem of Calculus.

$$\int_0^{1/2} \frac{1}{(1-x^2)^{1/2}}\,dx = \sin^{-1}(x)\Big|_0^{1/2} = \sin^{-1}(1/2) - \sin^{-1}(0) = \frac{\pi}{6}$$

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3. Using complete sentences and appropriate notation, state the Second Part of the Fundamental Theorem of Calculus.

Let $f(x)$ be a function that is continuous on an interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\,dt,$$

for each x in I, then $g'(x) = f(x)$ for each x in I.

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4. (a) Obtain the solution to the following initial value problem in terms of a definite integral taken with respect to the variable ***t***, so the differential denoting the variable of integration is ***dt***.

$$\frac{dy}{dx} = \sec^2(x)e^{\tan(x)} \quad \text{with } y(\pi/4) = 5$$

(b) Then by evaluating the *dt* integral, completely identify the solution, $y(x)$.

$$y(x) = 5 + \int_{\pi/4}^{x} \sec^2(t)e^{\tan(t)}\,dt \qquad (a)$$

$$= 5 + \int_1^{\tan(x)} e^u\,du = 5 + e^{\tan(x)} - e^1 \qquad (b)$$

$$= e^{\tan(x)} + 5 - e,\ \text{for } x \in (-\pi/2, \pi/2)$$

using the u-substitution $u = \tan(t)$.

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5. (a) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: Complete the sentence,"ln(x) = ...")

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

for $x > 0$. The domain of the natural log function is $(0,\infty)$, and its range is the whole real line, $(-\infty,\infty)$.

(b) Given $\ln(a) = -20$ and $\ln(c) = 5$, evaluate the following integral.

$$\int_a^{c^5} \frac{1}{\lambda}\,d\lambda = \ln(c^5) - \ln(a) = 25 - (-20) = 45.$$

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6. If the given integral is expressed as an equivalent integral in terms of the variable u using the substitution $u = x^2 - 1$ so that the equation below is true, what are the numerical values of the new limits of integration α and β, and what is the new integrand, $f(u)$??? Obtain these but do not attempt to evaluate the du integral.

$$\int_0^2 2x\,|x^2-1|^3\,dx = \int_\alpha^\beta f(u)\,du$$

$du = 2x\,dx$ $\qquad f(u) = |u|^3$

$\alpha = -1$ $\qquad \beta = 3$

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7. A particle moves with a velocity of $v(t) = t^2 - 1$ along an s-axis. Find the displacement and total distance traveled over the time interval [0,2].

$$\text{Displacement} = \int_0^2 v(t)\,dt = \int_0^2 t^2-1\,dt = \ldots = \frac{2}{3}$$

$$\text{Total\_Distance} = \int_0^2 |v(t)|\,dt = \int_0^2 |t^2-1|\,dt = \ldots = 2$$

[See Example 8, Section 6.6 of Anton's 8th.]

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8. Find each of the following derivatives.

(a) $$\frac{d}{dx}\left[\int_1^x \frac{\sin(t)}{t}\,dt\right] = \frac{\sin(x)}{x}$$

(b) $$\frac{d}{dx}\left[x^3 + \int_1^{\tan(x)} \tan^{-1}(t)\,dt\right] = 3x^2 + \tan^{-1}(\tan(x))\sec^2(x)$$

$$= 3x^2 + x\sec^2(x)$$

9. Evaluate the following limits:

(*a*) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

(*b*) $\lim_{x\to\infty}\left(1+\frac{1}{5x}\right)^{\ln(32)x} = \lim_{u\to\infty}\left[\left(1+\frac{1}{u}\right)^{u}\right]^{\ln(2)} = e^{\ln(2)} = 2$

by using the substitution $u = 5x$ plus some log magic.

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10. Let the function g be defined by the equation

$$g(x) = \int_1^x (3t^2+1)^{1/2}\,dt$$

for $x \in (-\infty,\infty)$. Then

(a) $g(1) = 0$.

Then since $g'(x) = (3x^2+1)^{1/2}$ and $g''(x) = \frac{3x}{(3x^2+1)^{1/2}}$,

(b) $g'(1) = 2$, *and*

(c) $g''(1) = \frac{3}{2}$.

(d) Determine the open intervals where g is increasing or decreasing. Be specific.

By considering the sign of g′ above, it is evident that g is increasing on all of $\mathbb{R} = (-\infty,\infty)$.//

(d) Determine the open intervals where g is concave up or concave down. Be specific.

By considering the sign of g″ above, we can see that g is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.//

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11. Find the exact arc length of the parametric curve without eliminating the parameter when $x = \cos(2t)$ and $y = \sin(2t)$ when $0 \le t \le \pi/2$.

$$L = \int_0^{\pi/2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} dt$$
$$= \int_0^{\pi/2}\left(4\sin^2(2t) + 4\cos^2(2t)\right)^{1/2} dt = \int_0^{\pi/2} 2\,dt = \pi$$

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12. A spring exerts a force of 5 N when stretched 1 m beyond its natural length. How much work is required to stretch the spring 1.8 m beyond its natural length?

$$\text{Work} = \int_0^{9/5} F(x)\,dx = \int_0^{9/5} 5x\,dx = \left(\frac{5}{2}x^2\right)\Big|_0^{9/5} = \frac{81}{10}\text{ J}$$ since $(1)k = 5$ implies

$k = 5$ N per meter using the standard Hooke’s model.

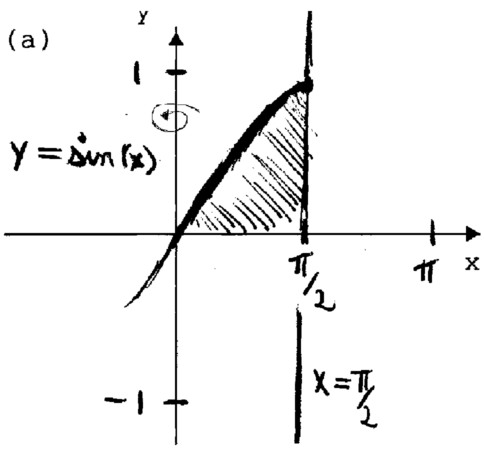
13. Evaluate the following integral.

$$\int x^2 e^x\,dx = x^2e^x - \int 2x\,e^x\,dx$$
$$= x^2e^x - \left[2x\,e^x - \int 2\,e^x\,dx\right]$$
$$= x^2e^x - 2x\,e^x + 2\,e^x + C$$

by integrating by parts twice, all the while picking on the exponential as the recognized derivative, that is , setting $dv = e^x\,dx$.

14. (a) Sketch the region in the 1st quadrant enclosed by the curves defined by y = sin x, y = 0, and x = π/2. Suppose the region is revolved around the y axis. (b) Using the method of cylindrical shells, compute the volume of the solid of revolution formed.

(a)



(b)

$$V = \int_0^{\pi/2} 2\pi x\sin(x)\,dx$$
$$= -(2\pi x\cos(x))\Big|_0^{\pi/2} + \int_0^{\pi/2} 2\pi\cos(x)\,dx$$
$$= 2\pi\int_0^{\pi/2}\cos(x)\,dx$$
$$= 2\pi\sin(x)\Big|_0^{\pi/2}$$
$$= 2\pi\sin\left(\frac{\pi}{2}\right) - 2\pi\sin(0) = 2\pi$$

by using integration by parts in the obvious way, setting $u = 2\pi x$ and $dv = \sin(x)\,dx$.

15. Evaluate the following integral.

$$\int \frac{1}{(1+x^2)^{1/2}}\,dx = \int \frac{\sec^2(\theta)}{(1+\tan^2(\theta))^{1/2}}\,d\theta$$
$$= \int \sec(\theta)\,d\theta$$
$$= \ln|\sec(\theta) + \tan(\theta)| + C$$
$$= \ln|(1+x^2)^{1/2} + x| + C$$

using the trig substitution $x = \tan(\theta)$.

16. Complete each trigonometric identity with an expression involving cos(2x).

(a) $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$

(b) $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$

17. Evaluate the integral.

$$\int e^{-x}\tan(e^{-x})\,dx = -\ln|\sec(e^{-x})| + C$$ using the u-substitution $u = e^{-x}$.

18. Evaluate the integral.

$$\int \sec(4x)\,dx = \frac{1}{4}\ln|\sec(4x)+\tan(4x)| + C$$ using the u-substitution $u = 4x$.

19. Evaluate the integral.

$$\int_1^e \ln(x)\,dx = x\ln(x)\Big|_1^e - \int_1^e x\left(\frac{1}{x}\right)dx$$

$$= e - \int_1^e 1\,dx = 1$$

doing the obvious integration by parts.

20. (a) Suppose that $n \geq 2$ is a positive integer. Show in detail how to derive the following reduction formula:

$$\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

First, factor $\cos^n(x)$, put on your sun glasses to protect yourself from the uv rays, and then do a simple integration by parts after choosing $u = \cos^{n-1}(x)$ and $dv = \cos(x)dx$. Then, magically, you have

$$\int\cos^n(x)dx = \int\cos^{n-1}(x)\cos(x)dx$$
$$= \cos^{n-1}(x)\sin(x) - (n-1)\int\cos^{n-2}(x)\sin^2(x)dx$$

since $du = (n-1)\cos^{n-2}(x)\sin(x)dx$ and $v = \sin(x)$.

By putting your favorite Pythagorean identity to work by replacing the $\sin^2(x)$ in the integral on the right side after the second equals sign above with $1 - \cos^2(x)$ and doing the obvious algebra and using the linearity of the integral, you may now produce

$$\int\cos^n(x)dx = \cos^{n-1}(x)\sin(x) - (n-1)\int\cos^n(x)dx + (n-1)\int\cos^{n-2}(x)dx$$

Finally, adding

$$(n-1)\int\cos^n(x)dx$$

to both sides of the equation above, and simplifying the left side algebraically, we have

$$n\int\cos^n(x)dx = \cos^{n-1}(x)\sin(x) + (n-1)\int\cos^{n-2}(x)dx$$

Multiplying by n^{-1} on both sides of this equation finishes the incantation.

(b) Use the reduction formula in Part (a) to evaluate the following definite integral:

$$\int_0^{\pi/2}\cos^4(x)\,dx = \frac{\cos^3(x)\sin(x)}{4}\Bigg|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2}\cos^2(x)\,dx$$
$$= \frac{3}{4}\int_0^{\pi/2}\cos^2(x)\,dx$$
$$= \frac{3}{4}\left[\frac{\cos(x)\sin(x)}{2}\Bigg|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos^0(x)\,dx\right]$$
$$= \frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2}1\,dx = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{16}.$$