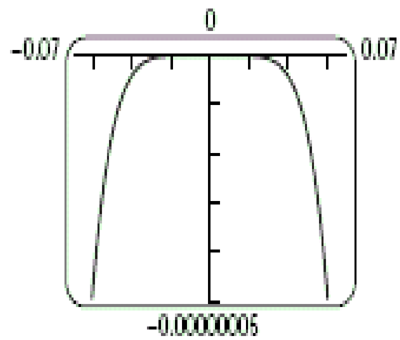
Anton, 9th Edition

Problem 45 of Section 9.7

First, if you are an instructor, don't assign this problem. Conceptually it is easy, but the details are horrendous.

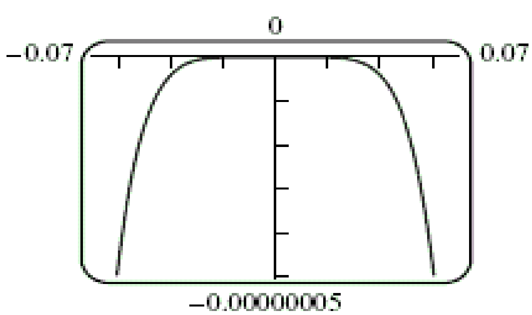
45. $f^{(5)}(x) = -\dfrac{3840x^5}{(1+x^2)^6} + \dfrac{3840x^3}{(1+x^2)^5} - \dfrac{720x}{(1+x^2)^4}$,
Assume first that $|x| < 1/2$, then
$|f^{(5)}(x)| < 3840|x|^5 + 3840|x|^3 + 720|x|$,
so let $M = 960$,
$R_4(x) \leq \dfrac{960}{4!}|x|^4 < 0.0005$ if $x < 0.059$



You will find above the answer provided in the Instructor's Solution Manual of the 9th edition. That might be viewed as an improvement over the solution in the 8th Edition Instructor's Solution Manual of what is effectively the same problem, Problem 41 of Section 10.7. Its solution is below:

41. $f^{(5)}(x) = -\dfrac{3840}{(1+x^2)^7} + \dfrac{3840x^3}{(1+x^2)^5} - \dfrac{720x}{(1+x^2)^4}$, let $M = 8400$,

$R_4(x) \leq \dfrac{8400}{4!}|x|^4 < 0.0005$ if $x < 0.0677$



Both are obviously flawed, for plainly

$$R_4(x) \leq \frac{M}{4!}|x|^4$$

with the given M appearing at the bottom of both answers should be

$$|R_4(x)| \le \frac{M}{5!}|x|^5 .$$

This means that the obviously flawed conditions on x that follow the "if" also are likely not on the mark.

Let's recall what the problem actually asks us to do.

Problem. Use the Remainder Estimation Theorem to find an interval containing $x = 0$ over which

$$f(x) = \frac{1}{1 + x^2}$$

can be approximated by $p(x) = 1 - x^2 + x^4$ to three decimal-place accuracy throughout the interval. You are then asked to check by using a graphing utility to graph $|f(x) - p(x)|$ over the interval you obtained.

A Sketch of the Key Ideas of the Solution:

In principle, solving the problem requires that we identify the given polynomial $p(x)$ as an appropriate Maclaurin polynomial of the function $f(x)$, that is as $p_{n_0}(x)$ for some nonnegative integer n_0. Then by choosing a suitable initial interval I_0, symmetric with respect to zero, we should obtain an upper bound M on the function $|f^{(n_0+1)}(x)|$ when $x \in I_0$. The Remainder Estimation Theorem then gives us an upper bound on the magnitude of the true error when $x \in I_0$, namely

$$|f(x) - p(x)| = |R_{n_0}(x)| \le \frac{M}{(n_0+1)!}|x|^{n_0+1}$$

since $p(x) = p_{n_0}(x)$. To then obtain the desired accuracy, it suffices to have

$$\frac{M}{(n_0+1)!}|x|^{n_0+1} < \frac{1}{2}10^{-3} .$$

This last inequality is an interval in disguise since it is equivalent to

$$|x| < \left[\frac{(n_0+1)!}{2M}10^{-3} \right]^{\frac{1}{n_0+1}} .$$

To finish up, then, we need only take the intersection of the interval defined by this inequality and the interval I_0 which allowed us to find the number M needed in applying the Remainder Estimation Theorem. This will provide us with an interval where we can prove that the polynomial $p(x)$ provides the desired approximation.

The Demon in the Details:

Unlike the other three problems in this set, where it is easy and routine to obtain derivatives of arbitrary order for the given function $f(x)$ for the problem, the function

$$f(x) = \frac{1}{1 + x^2}$$

presents a special challenge. The higher order derivatives are rational functions that become messier and messier to handle as the order of differentiation goes up. In fact, one might characterize the situation as one where the problem is one of differential drudgery.

To alleviate that drudgery somewhat, we shall make things complex, so as, ultimately to simplify them. Evidently,

$$f(x) = \frac{1}{(x + i)(x - i)},$$

where $i^2 = -1$. By performing a partial fraction decomposition using complex number coefficients, we may then write

$$f(x) = \frac{1}{2i}\left[\frac{1}{(x - i)} - \frac{1}{(x + i)} \right].$$

And why would we want to do something as outlandish as this?? It turns out that when

$$g(x) = \frac{1}{x - a},$$

by taking three or four derivatives, one can quickly guess that

$$g^{(k)}(x) = (-1)^k k!\,(x - a)^{-(k+1)}$$

for each integer $k \geq 0$. This formula remains valid when x and a are complex-valued, provided $x \neq a$.

Thus, by using the structure of $f(x)$, properties of the derivative, and routine algebra, we may write

$$f^{(k)}(x) = \frac{(-1)^k k!}{2i}\left[\frac{(x+i)^{k+1} - (x-i)^{k+1}}{(x^2 + 1)^{k+1}} \right]$$

for each integer $k \geq 0$ with the obvious restriction for complex values of x.

Using the Binomial Theorem and easily done algebra allows us to reveal what $(x+i)^{k+1} - (x-i)^{k+1}$ is as a polynomial with complex coefficients. In fact,

$$(x + i)^{k+1} - (x - i)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} x^{k+1-j} (i)^{j} \left[1 - (-1)^{j} \right]$$

for each integer $k \geq 0$. We may re-write the right hand side since

$$1 - (-1)^{j} = \begin{cases} 0 \quad , \quad if \;\; j \;\; is \;\; even \\ 2 \quad , \quad if \;\; j \;\; is \;\; odd\,. \end{cases}$$

We can see from this that the only nonzero terms are those where j is odd and no larger than $k+1$. Since the number of odd positive integers no larger than $k+1$ is given using the greatest integer function by

$$O\,(k+1) = \left[\!\left[\frac{k+2}{2} \right]\!\right],$$

we may write

$$(x + i)^{k+1} - (x - i)^{k+1} = \sum_{j=1}^{O\,(k+1)} 2 \binom{k+1}{2j-1} x^{k+1-(2j-1)} (i)^{2j-1} \,.$$

Doing a little more algebra provides us with

$$(x + i)^{k+1} - (x - i)^{k+1} = (2i) \sum_{j=1}^{O\,(k+1)} (-1)^{j+1} \binom{k+1}{2j-1} x^{k+2-2j} \,.$$

Finally we may put together the pieces to obtain

$$f^{\,(k)}(x) = \frac{(-1)^{k}\, k!}{(x^{2} + 1)^{k+1}} \left[\sum_{j=1}^{O\,(k+1)} (-1)^{j+1} \binom{k+1}{2j-1} x^{k+2-2j} \right]$$

where $O\,(k+1)$ is the number of odd positive integers no larger than $k+1$ for $k \geq 0$.

It is convenient now to explicitly write out the first six derivatives of

$$f\,(x) = \frac{1}{1 + x^{2}}\,,$$

so that we can deal with the details of Maclaurin polynomials and an upper bound on the remainder term. Thus, either using the formula above or two and a half pages of computations using logarithmic differentiation, you can obtain the following:

$$f^{(1)}(x) = \frac{(-1)2x}{(x^2 + 1)^2};$$

$$f^{(2)}(x) = \frac{(-1)^2\, 2!}{(x^2 + 1)^3}\left[3x^2 - 1 \right];$$

$$f^{(3)}(x) = \frac{(-1)^3\, 3!}{(x^2 + 1)^4}\left[4x^3 - 4x \right];$$

$$f^{(4)}(x) = \frac{(-1)^4\, 4!}{(x^2 + 1)^5}\left[5x^4 - 10x^2 + 1 \right];$$

$$f^{(5)}(x) = \frac{(-1)^5\, 5!}{(x^2 + 1)^6}\left[6x^5 - 20x^3 + 6x \right];$$

and

$$f^{(6)}(x) = \frac{(-1)^6\, 6!}{(x^2 + 1)^7}\left[7x^6 - 35x^4 + 21x^2 - 1 \right].$$

Consequently, we have $f^{(1)}(0) = f^{(3)}(0) = f^{(5)}(0) = 0$, $f^{(2)}(0) = -2!$, and $f^{(4)}(0) = 4!$. Thus, we may view the polynomial

$$p(x) = 1 - x^2 + x^4$$

as either the 4th or the 5th Maclaurin polynomial of f.

We should hope that the higher order polynomial will give a better error bound since

$$\left[\frac{1}{2}10^{-3} \right]^{\frac{1}{5}} < \left[\frac{1}{2}10^{-3} \right]^{\frac{1}{6}}.$$

Of course the fifth or sixth derivative of the function also plays a part in determining the final interval, as does the initial interval on which we study the appropriate derivative. For definiteness, now, we shall take $n_0 = 5$ and use nearly the same interval as the Instructor's Solution Manual, the set I_0 of real numbers x with $|x| \leq 1/2$.

It really is not feasible to easily obtain the absolute extrema of the sixth derivative, or even of its polynomial factor $7x^6 - 35x^4 + 21x^2 - 1$ with x restricted to the closed interval I_0. What we shall do instead is use the triangle inequality to obtain a crude upper bound. Plainly,

$$|\,7x^6 - 35x^4 + 21x^2 - 1\,| \le 7|x|^6 + 35|x|^4 + 21|x|^2 + 1$$

and

$$7|x|^6 + 35|x|^4 + 21|x|^2 + 1 \le \frac{7}{64} + \frac{35}{16} + \frac{21}{4} + 1 < 10$$

when $|x| < 1/2$. Thus, if $|x| < 1/2$, $\left|f^{(6)}(x)\right| \le (10)\,6!$. Consequently we may take $M = (10)6!$.

You may wish to refer back to the sketch of the key ideas that appeared earlier. To obtain the desired accuracy, it now suffices to have

$$|x| < \left[\ \frac{6!}{2(10)6!}10^{-3}\ \right]^{\frac{1}{6}} = \left[\ \frac{1}{2}\,10^{-4}\ \right]^{\frac{1}{6}}.$$

Observe that a calculator provides

$$[\ \frac{1}{2}\,10^{-4}\ ]^{\frac{1}{6}} \approx 0.1919383\,.$$

Finally, it is worth noting that if one choses to disregard the requirement that the Remainder Estimation Theorem be used, it is easy to obtain an interval where the polynomial $p(x) = 1 - x^2 + x^4$ approximates

$$f(x) = \frac{1}{1 + x^2}$$

to three decimal places. The lucky circumstance is that the polynomial is a geometric sum. Thus, as long as x is a real number, we have

$$|\,f(x) - p(x)| = \left|\frac{1}{1 + x^2} - \left(\frac{1 - (-x^2)^3}{1 + x^2}\right)\right| = \frac{|\,x\,|^6}{1 + x^2} \le |\,x\,|^6.$$

To obtain the desired accuracy, it suffices to have $|\,x\,| < [\,(1/2)\,10^{-3}\,]^{1/6}$. Again resorting to a calculator, we can see that $[\,(1/2)\,10^{-3}\,]^{1/6} \approx 0.28172691$. That's better?? // eM toidI