READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds. Eschew obfuscation.*

1. (5 pts.) (a) On the first line write the expression using a definite integral that would be used to compute the average value of f(x) = sin(x) over the interval [π/3,π]. (b) Then below evaluate the expression to obtain the numerical value of the average.

(a) $$f_{AVE} = \frac{1}{\pi - (\pi/3)} \int_{\pi/3}^{\pi} \sin(x)\ dx$$

(b) $$= \frac{3}{2\pi}\left[(-\cos(x))\Big|_{\pi/3}^{\pi}\right] = \frac{3}{2\pi}\left[1 + \frac{1}{2}\right] = \frac{9}{4\pi}$$

2. (5 pts.) Using a complete sentence and appropriate notation, state precisely the First Part of the Fundamental Theorem of Calculus. [This is sometimes called the *Evaluation Theorem*.]

If $f(x)$ is continuous on $[a,b]$ and $g(x)$ is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\ dx = g(b) - g(a).$$

3. (5 pts.) Express the following sum using sigma notation, but do not attempt to find its numerical value.

$$-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \frac{1}{8^2} - \frac{1}{10^2} + \frac{1}{12^2} = \sum_{k=1}^{6} \frac{(-1)^k}{(2k)^2}$$

4. (5 pts.) Obtain the upper limit of summation and rewrite the summand in order to make the following equation true. **Do not attempt to evaluate the sum.**

$$\sum_{k=0}^{45} 2^{3k} = \sum_{j=8}^{53} 2^{3(j-8)}$$

since $j = k + 8$ is equivalent to $k = j - 8$.

5. (5 pts.) Express the following limit as a definite integral. Do not attempt to evaluate the integral.

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (2\sin(e^{x_k^*}))\Delta x_k\ ; \quad a = -\pi,\ b = 2.$$

$$L = \int_{-\pi}^{2} 2\sin(e^x)\ dx$$

6. (5 pts.) Using complete sentences and appropriate notation, state the Second Part of the Fundamental Theorem of Calculus.

// Let f(x) be a function that is continuous on an interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\ dt,$$

for each x in I, then g′(x) = f(x) for each x in I.//

7. (10 pts.) Find each of the following derivatives.

(a) $\dfrac{d}{dx}\left[\int_x^{\pi} 4\cos(e^{t^3})\ dt\right] = -4\cos(e^{x^3})$

(b) $\dfrac{d}{dx}\left[\int_1^{x^2} \cos^3(t)\ dt\right] = \cos^3(x^2)\,2x = 2x\cos^3(x^2)$

8. (5 pts.) (a) (3 pts.) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: To start, complete the sentence,"ln(x) = ...")

$$\ln(x) = \int_1^x \frac{1}{t}\ dt$$

for $x > 0$. The domain of the natural log function is $(0,\infty)$, and its range is the whole real line, $(-\infty,\infty)$.

(b) (2 pts.) Given a and b are positive numbers with ln(a) = 4 and ln(b) = -3, evaluate the following integral.

$$\int_{b^2}^{a} \frac{1}{t}\ dt = \ln(a) - \ln(b^2) = \ln(a) - 2\ln(b) = 10$$

9. (5 pts.) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable **t**, so the differential denoting the variable of integration is ***dt***. DO NOT ATTEMPT TO EVALUATE THE DEFINITE INTEGRAL IN t THAT YOU OBTAIN.

$$\frac{dy}{dx} = 4\sin(x^2), \quad y(\pi) = 1.$$

$$y(x) = 1 + \int_{\pi}^{x} 4\sin(t^2)\ dt$$

10. (5 pts.) Evaluate the definite integral below by expressing it in terms of u correctly and then evaluating the resulting integral using a formula from geometry.

$$\int_{\tan^{-1}(-9)}^{0} \sqrt{81-\tan^2(x)}\, 4\sec^2(x)\ dx\ ;\ u = \tan(x)$$

$$\int_{\tan^{-1}(-9)}^{0} \sqrt{81-\tan^2(x)}\, 4\sec^2(x)\ dx = 4\int_{-9}^{0} \sqrt{81-u^2}\ du = 81\pi$$

The right-most integral provides the area of a quarter circle with a radius of 9 centered at the origin.

11. (10 pts.) Write each of the following two sums in closed form.

(a) $\displaystyle\sum_{k=0}^{49}\left(\frac{1}{2^k}\right) = \frac{1-(1/2)^{50}}{1-(1/2)} = 2\left[1-\left(\frac{1}{2}\right)^{50}\right]$

(b) $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+3}\right) = \frac{1}{3}-\frac{1}{n+3}$

12. (10 pts.) If the function f is continuous on [a,b], then the *net signed area* A between $y = f(x)$ and the interval [a,b] is defined by

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x.$$

Reveal all the details in computing the numerical value of the net signed area of $f(x) = 16x^3$ over the interval [0,1] using only the definition above with

$$x_k^*$$

the right end point of each subinterval in the regular partition. *Do not use the Fundamental Theorem, Part 1.*

$$\begin{aligned} A &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} 16\left(\frac{k}{n}\right)^3\frac{1}{n} \\ &= \lim_{n\to\infty}\left(\frac{16}{n^4}\right)\sum_{k=1}^{n} k^3 = \lim_{n\to\infty}\left(\frac{16n^2(n+1)^2}{4n^4}\right) \\ &= \lim_{n\to\infty} 4\left(1+\frac{1}{n}\right)^2 = 4 \end{aligned}$$

since $\Delta x = 1/n$ and $x_k = k/n$ for $k = 0, 1, \ldots, n$ are the end points of the intervals of the general regular partition.

13. (10 pts.) A particle moves with a velocity of $v(t) = 4 - 2t$ along an s-axis. Find the displacement and total distance traveled over the time interval [0,3].

$$\text{Displacement} = \int_0^3 4-2t\,dt = (4t-t^2)\Big|_0^3 = (12-9) - (0-0) = 3$$

$$\begin{aligned}\text{Total\_Distance} &= \int_0^3 |4-2t|\,dt \\ &= \int_0^2 |4-2t|\,dt + \int_2^3 |4-2t|\,dt \\ &= \int_0^2 4-2t\,dt + \int_2^3 2t-4\,dt = (4t-t^2)\Big|_0^2 + (t^2-4t)\Big|_2^3 \\ &= (4-0)+(-3-(-4)) = 5\end{aligned}$$

14. (5 pts.) Evaluate the following limit:

$$L = \lim_{x\to\infty}\left(1 + \frac{\ln(3)}{x}\right)^x = \lim_{u\to\infty}\left[\left(1+\frac{1}{u}\right)^u\right]^{\ln(3)} = e^{\ln(3)} = 3$$

using the substitution $1/u = \ln(3)/x$.

15. (10 pts.) Let the function g be defined by the equation

$$g(x) = \int_0^x \tan^{-1}(t)\,dt + \tfrac{\pi}{2}x$$

for $x \in (-\infty, \infty)$. Then since

$$g'(x) = \tan^{-1}(x) + \frac{\pi}{2} \quad \text{and} \quad g''(x) = \frac{1}{1+x^2} ,$$

(a) $g(0) = 0$

(b) $g'(0) = \frac{\pi}{2}$

(c) $g''(0) = 1$

(d) Determine the open intervals where g is increasing or decreasing. Be specific.

Since $g'(x) = \tan^{-1}(x) - (-\pi/2)$, *and* $\tan^{-1}(x) > -\pi/2$ *for every* x, $g'(x) > 0$ *for every* x. *Thus,* g *is increasing on* $\mathbb{R} = (-\infty, \infty)$.

(e) Determine the open intervals where g is concave up or concave down. Be specific.

Since $g''(x) > 0$ *for every* x, g *is concave up on* $\mathbb{R} = (-\infty, \infty)$.

Silly 10 Point Bonus: State the *Mean-Value Theorem for Integrals* and then use it to prove *Part 2 of the Fundamental Theorem of Calculus*.
[Say where your work is, for it won't fit here.]