READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page, for I do not read minds. Eschew obfuscation.*

1. (5 pts.) (a) On the first line write the expression using a definite integral that would be used to compute the average value of f(x) = sin(x) over the interval [π/6,π]. (b) Then below evaluate the expression to obtain the numerical value of the average.

(a) $f_{AVE} = \dfrac{1}{\pi - (\pi/6)} \int_{\pi/6}^{\pi} \sin(x)\ dx$

(b) $= \dfrac{6}{5\pi}\left[(-\cos(x))\Big|_{\pi/6}^{\pi}\right] = \dfrac{6}{5\pi}\left[1 + \dfrac{\sqrt{3}}{2}\right] = \dfrac{3}{5\pi}\left[2 + \sqrt{3}\right]$

2. (5 pts.) Using a complete sentence and appropriate notation, state precisely the First Part of the Fundamental Theorem of Calculus. [This is sometimes called the *Evaluation Theorem*.]

If $f(x)$ is continuous on $[a,b]$ and $g(x)$ is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\ dx = g(b) - g(a).$$

3. (5 pts.) Express the following sum using sigma notation, but do not attempt to find its numerical value.

$$-\frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} = \sum_{k=1}^{6} \frac{(-1)^k}{(2k+1)^3}$$

4. (5 pts.) Obtain the upper limit of summation and rewrite the summand in order to make the following equation true. **Do not attempt to evaluate the sum.**

$$\sum_{k=0}^{45} 3^{2k} = \sum_{j=10}^{55} 3^{2(j-10)}$$

since $j = k + 10$ is equivalent to $k = j - 10$.

5. (5 pts.) Express the following limit as a definite integral. Do not attempt to evaluate the integral.

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (\cos(e^{x_k^*}))\Delta x_k \ ; \quad a = -2,\ b = \pi.$$

$$L = \int_{-2}^{\pi} \cos(e^x)\ dx$$

6. (5 pts.) Using complete sentences and appropriate notation, state the Second Part of the Fundamental Theorem of Calculus.

// Let f(x) be a function that is continuous on an interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\ dt,$$

for each x in I, then g′(x) = f(x) for each x in I.//

7. (10 pts.) Find each of the following derivatives.

(a) $\frac{d}{dx}\left[\int_x^\pi 4\sin(e^{t^3})\ dt \right] = -4\sin(e^{x^3})$

(b) $\frac{d}{dx}\left[\int_1^{x^3} \sin^2(t)\ dt \right] = \sin^2(x^3)3x^2 = 3x^2\sin^2(x^3)$

8. (5 pts.) (a) (3 pts.) Give the definition of the function ln(x) in terms of a definite integral and give its domain and range. Label correctly. (**Hint**: To start, complete the sentence,"ln(x) = ...")

$$\ln(x) = \int_1^x \frac{1}{t}\ dt$$

for $x > 0$. The domain of the natural log function is $(0,\infty)$, and its range is the whole real line, $(-\infty,\infty)$.

(b) (2 pts.) Given a and b are positive numbers with ln(a) = 8 and ln(b) = -3, evaluate the following integral.

$$\int_a^{b^2} \frac{1}{t}\ dt = \ln(b^2) - \ln(a) = 2\ln(b) - \ln(a) = -14$$

9. (5 pts.) Write the solution to the following initial value problem in terms of a definite integral taken with respect to the variable **t**, so the differential denoting the variable of integration is ***dt***. DO NOT ATTEMPT TO EVALUATE THE DEFINITE INTEGRAL IN t THAT YOU OBTAIN.

$$\frac{dy}{dx} = 4\cos(x^3), \quad y(1) = \pi.$$

$$y(x) = \pi + \int_1^x 4\cos(t^3)\ dt$$

10. (5 pts.) Evaluate the definite integral below by expressing it in terms of u correctly and then evaluating the resulting integral using a formula from geometry.

$$\int_{\tan^{-1}(-7)}^{0} \sqrt{49-\tan^2(x)}\,4\sec^2(x)\;dx\;;\;u = \tan(x)$$

$$\int_{\tan^{-1}(-7)}^{0} \sqrt{49-\tan^2(x)}\,4\sec^2(x)\;dx \;=\; 4\int_{-7}^{0}\sqrt{49-u^2}\;du \;=\; 49\pi$$

The right-most integral provides the area of a quarter circle with a radius of 7 centered at the origin.

11. (10 pts.) Write each of the following two sums in closed form.

$$(a)\quad \sum_{k=0}^{49}\left(\frac{1}{4^k}\right) = \frac{1-(1/4)^{50}}{1-(1/4)} = \frac{4}{3}\left[1-\left(\frac{1}{4}\right)^{50}\right]$$

$$(b)\quad \sum_{k=1}^{n}\left(\frac{1}{k+3}-\frac{1}{k+4}\right) = \frac{1}{4}-\frac{1}{n+4}$$

12. (10 pts.) If the function f is continuous on [a,b], then the *net signed area* A between $y = f(x)$ and the interval [a,b] is defined by

$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x.$$

Reveal all the details in computing the numerical value of the net signed area of $f(x) = 12x^3$ over the interval [0,1] using only the definition above with

$$x_k^*$$

the right end point of each subinterval in the regular partition. *Do not use the Fundamental Theorem, Part 1.*

$$\begin{aligned} A &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} 12\left(\frac{k}{n}\right)^3\frac{1}{n} \\ &= \lim_{n\to\infty}\left(\frac{12}{n^4}\right)\sum_{k=1}^{n} k^3 = \lim_{n\to\infty}\left(\frac{12n^2(n+1)^2}{4n^4}\right) \\ &= \lim_{n\to\infty} 3\left(1+\frac{1}{n}\right)^2 = 3 \end{aligned}$$

since $\Delta x = 1/n$ and $x_k = k/n$ for $k = 0, 1, \ldots, n$ are the end points of the intervals of the general regular partition.

13. (10 pts.) A particle moves with a velocity of $v(t) = 4 - 4t$ along an s-axis. Find the displacement and total distance traveled over the time interval [0,3].

$$\text{Displacement} = \int_0^3 4-4t\,dt = (4t-2t^2)\Big|_0^3 = (12-18) - (0-0) = -6$$

$$\begin{aligned}\text{Total\_Distance} &= \int_0^3 |4-4t|\,dt \\ &= \int_0^1 |4-4t|\,dt + \int_1^3 |4-4t|\,dt \\ &= \int_0^1 4-4t\,dt + \int_1^3 4t-4\,dt = (4t-2t^2)\Big|_0^1 + (2t^2-4t)\Big|_1^3 \\ &= (2-0)+(6-(-2)) = 10\end{aligned}$$

14. (5 pts.) Evaluate the following limit:

$$L = \lim_{x\to\infty}\left(1 + \frac{\ln(6)}{x}\right)^x = \lim_{u\to\infty}\left[\left(1 + \frac{1}{u}\right)^u\right]^{\ln(6)} = e^{\ln(6)} = 6$$

using the substitution $1/u = \ln(6)/x$.

15. (10 pts.) Let the function g be defined by the equation

$$g(x) = \int_0^x 2\tan^{-1}(t)\,dt - \pi x$$

for $x \in (-\infty, \infty)$. Then since

$$g'(x) = 2\tan^{-1}(x) - \pi \quad \text{and} \quad g''(x) = \frac{2}{1+x^2},$$

(a) $g(0) = 0$

(b) $g'(0) = -\pi$

(c) $g''(0) = 2$

(d) Determine the open intervals where g is increasing or decreasing. Be specific.

Since $g'(x) = 2[\tan^{-1}(x) - (\pi/2)]$, *and* $\tan^{-1}(x) < \pi/2$ *for every* x, $g'(x) < 0$ *for every* x. *Thus,* g *is decreasing on* $\mathbb{R} = (-\infty, \infty)$.

(e) Determine the open intervals where g is concave up or concave down. Be specific.

Since $g''(x) > 0$ *for every* x, g *is concave up on* $\mathbb{R} = (-\infty, \infty)$.

Silly 10 Point Bonus: State the *Mean-Value Theorem for Integrals* and then use it to prove *Part 2 of the Fundamental Theorem of Calculus*.
[Say where your work is, for it won't fit here.]