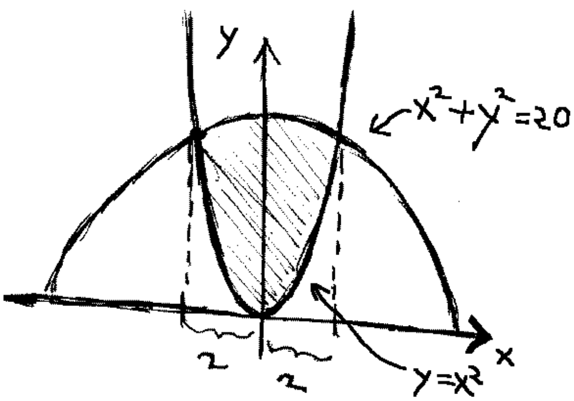
READ ME FIRST: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Eschew obfuscation. Show me all the magic on the page.*

1. (25 pts.)
The region R bounded between the curves $x^2 + y^2 = 20$ and $y = x^2$ is sketched below for your convenience. *The sketch does not have all the information you need for this problem!!*



(a) Write down, but do not attempt to evaluate the definite integral whose numerical value gives the area of the region R if one integrates with respect to *x* so the differential in the integral is *dx*.

$$\text{Area} = \int_{-2}^{2} \sqrt{20-x^2} - x^2\, dx$$

(b) Write down, but do not attempt to evaluate the sum of definite integrals whose numerical value gives the area of the region R if one integrates with respect to *y* so the differential in the integral is *dy*.

$$\text{Area} = \int_0^4 \sqrt{y} - (-\sqrt{y})\, dy + \int_4^{2\sqrt{5}} \sqrt{20-y^2} - (-\sqrt{20-y^2})\, dy$$

(c) A first step to evaluate the definite integral in part (a) above is to perform a suitable trigonometric substitution. Explicitly give the substitution and provide the definite integral with respect to θ that results, but do not attempt to evaluate the *d*θ integral you have obtained.

$$\textit{Using } x = \sqrt{20}\sin(\theta) \textit{ so that } dx = \sqrt{20}\cos(\theta)\, d\theta \textit{ and } \theta = \sin^{-1}\left(\frac{x}{\sqrt{20}}\right),$$

$$\text{Area} = \int_{\sin^{-1}(-1/\sqrt{5})}^{\sin^{-1}(1/\sqrt{5})} \left[\sqrt{20 - 20\sin^2(\theta)} - 20\sin^2(\theta)\right]\sqrt{20}\cos(\theta)\, d\theta$$

(d) Write down, but do not attempt to evaluate, the definite integral that provides the numerical value of the arc-length of the lower curve above, $y = x^2$ from $x = -2$ to $x = 2$.

$$\text{Length} = \int_{-2}^{2} \sqrt{1+(2x)^2}\, dx$$

(e) A first step to evaluate the definite integral in part (d) above is to perform a suitable trigonometric substitution. Explicitly give the substitution and provide the definite integral with respect to θ that results, but do not attempt to evaluate the *d*θ integral you have obtained.

$$\textit{Using } 2x = \tan(\theta) \textit{ so that } dx = \frac{1}{2}\sec^2(\theta)\, d\theta \textit{ and } \theta = \tan^{-1}(2x),$$

$$\text{Length} = \int_{\tan^{-1}(-4)}^{\tan^{-1}(4)} \sqrt{1 + \tan^2(\theta)}\, \frac{1}{2}\sec^2(\theta)\, d\theta$$

2. (5 pts.) Suppose that $n \geq 2$ is a positive integer. By using integration by parts and an appropriate trigonometric identity, show in detail how to derive the following reduction formula:

$$\int \cos^{n}(x)\ dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\ dx$$

First, factor $\cos^n(x)$, put on your sun glasses to protect yourself from the uv rays, and then do a simple integration by parts after choosing $u = \cos^{n-1}(x)$ and $dv = \cos(x)dx$. Then, magically, you have

$$\begin{aligned}\int \cos^{n}(x)\ dx &= \int \cos^{n-1}(x)\cos(x)\ dx\\ &= \cos^{n-1}(x)\sin(x) - (n-1)\int \cos^{n-2}(x)\sin^2(x)\ dx\end{aligned}$$

since $du = (n-1)\cos^{n-2}(x)\sin(x)dx$ and $v = \sin(x)$.

By putting your favorite Pythagorean identity to work by replacing the $\sin^2(x)$ in the integral on the right side after the second equals sign above with $1 - \cos^2(x)$ and doing the obvious algebra and using the linearity of the integral, you may now produce

$$\int \cos^{n}(x)\ dx = \cos^{n-1}(x)\sin(x) - (n-1)\int \cos^{n}(x)\ dx + (n-1)\int \cos^{n-2}(x)\ dx$$

Finally, adding $(n-1)\int \cos^{n}(x)\ dx$ to both sides of the equation above, and simplifying the left side algebraically, we have

$$n\int \cos^{n}(x)\ dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\ dx$$

Multiplying by n^{-1} on both sides of this equation finishes the incantation.

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3. (20 pts.) (a) (5 pts.) Using literal constants A, B, C, etc., write the form of the partial fraction decomposition for the proper fraction below. Do not attempt to obtain the actual numerical values of the constants A, B, C, etc. Be very careful here.

$$\frac{3x^2+5}{x(x-2)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} + \frac{Gx+H}{(x^2+1)^3}$$

(b) (5 pts.) Obtain the numerical values of the literal constants A, B, and C in the partial fraction decomposition below.

$$(*) \qquad \frac{x-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

A, B, and C satisfy (*) for each x different from zero if, and only

$$(A+B)x^2 + Cx + A = 0x^2 + 1x - 2$$

for every real number x. Equating coefficients and solving the resulting linear system results in $A = -2$, $B = 2$, and $C = 1$.//

(c) (10 pts.) If one were to integrate the rational function in part (a), one might also encounter the integral below. Evaluate the integral below.

$$\begin{aligned}\int \frac{1}{(x^2+1)^3}\ dx &= \int \frac{\sec^2(\theta)\ d\theta}{\sec^6(\theta)} = \int \cos^4(\theta)\ d\theta\\ &= \frac{\cos^3(\theta)\sin(\theta)}{4} + \frac{3\cos(\theta)\sin(\theta)}{8} + \frac{3\theta}{8} + C\\ &= \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3\tan^{-1}(x)}{8} + C\end{aligned}$$

using the trigonometric substitution $x = \tan(\theta)$ and the reduction formula given in Problem 2 twice. [You get to fill in that lacuna.]

4. (50 pts.) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals. Be careful, for some of the definite integrals are improper. To get any credit, they must be handled correctly!!! [5 pts./part]

(a) $$\int_0^{\sqrt{\ln(7)}} 6x\,e^{x^2}\,dx = \int_0^{\ln(7)} 3e^u\,du = (3e^u)\Big|_0^{\ln(7)} = 3e^{\ln(7)} - 3e^0 = 18$$

by using the u-substitution $u = x^2$.

(b)
$$\int_0^{\infty} x e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x e^{-x}\,dx = \lim_{b\to\infty}(-xe^{-x} - e^{-x})\Big|_0^b$$
$$= \lim_{b\to\infty}\left(1 - \frac{1}{e^b} - \frac{b}{e^b}\right) = 1$$

by integrating by parts using $u = x$ and $dv = \exp(-x)dx$.

(c)
$$\int_0^{\pi} 16\sin^2(t)\,dt = \int_0^{\pi} 16\left[\frac{1-\cos(2t)}{2}\right]dt$$
$$= \int_0^{\pi} 8-8\cos(2t)\,dt = (8t-4\sin(2t))\Big|_0^{\pi}$$
$$= 8\pi - 4\sin(2\pi) - (0-\sin(0)) = 8\pi$$

via trig or treat.

(d)
$$\int_0^{\pi/4} 2\tan(2x)\,dx = \lim_{b\to\pi/4^-}\int_0^b 2\tan(2x)\,dx = \lim_{b\to\pi/4^-}\ln(\sec(2x))\Big|_0^b$$
$$= \lim_{b\to\pi/4^-}\ln(\sec(2b)) = +\infty$$

using the u-substitution $u = 2x$.

(e) $$\int \frac{8t}{\sqrt{1+4t^2}}\,dt = 2\sqrt{1+4t^2} + C$$

using the u-substitution $u = 1 + 4t^2$.

(f)
$$\int \frac{\sin^3(x)}{\cos(x)}\,dx = \int \frac{\sin(x)(1-\cos^2(x))}{\cos(x)}\,dx$$
$$= \int \tan(x)\,dx - \int \sin(x)\cos(x)\,dx$$
$$= \ln|\sec(x)| - \frac{1}{2}\sin^2(x) + C$$

This may be handled in a variety of ways in the end so that there are many equivalent ending forms. For instance, above, after using the obvious trigonometric identity, you might do the substitution $u = \cos(x)$, etc.

(g)
$$\int_0^1 \frac{2x+1}{x^2+1}\,dx = \int_0^1 \frac{2x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx = (\ln(x^2+1))\Big|_0^1 + (\tan^{-1}(x))\Big|_0^1$$
$$= \ln(2) - \ln(1) + \tan^{-1}(1) - \tan^{-1}(0) = \ln(2) + \frac{\pi}{4}$$

This may also be handled using the trigonometric substitution $x = \tan(\theta)$ easily.

4. (Cont.) Evaluate each of the following antiderivatives or definite integrals. Give exact values for definite integrals. Be careful, for some of the definite integrals are improper. To get any credit, they must be handled correctly!!!
[5 pts./part] (h)

$$\int x^2\cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx$$

$$= x^2\sin(x) - \left[(2x)\cdot(-\cos(x)) - \int 2(-\cos(x))\,dx\right]$$

$$= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

by integrating by parts twice, all the while picking the trigonometric factor as the recognized derivative.

(i)

$$\int \frac{4}{x^2-x}\,dx = \int \frac{4}{x(x-1)}\,dx = \int \frac{4}{x-1} - \frac{4}{x}\,dx = 4\ln\left|\frac{x-1}{x}\right| + C$$

after doing an easy and obvious partial-fraction decomposition.

(j) $$\int \frac{2x^2+1}{x+2}\,dx = \int 2x - 4 + \frac{9}{x+2}\,dx = x^2 - 4x + 9\ln|x+2| + C$$

after doing an obvious long division.

Silly 10 Point Bonus: (a) Prove that

(*) $$\frac{1}{t} \geq \frac{4}{3} - \frac{4t}{9}$$

for every t ε [1,2]. (b) Using (a) to compare a couple of integrals, prove ln(2) > 2/3. //Say where your work is, for there isn't room here.

(a) By doing elementary algebra, the truth of inequality (*) above on the interval [1,2] may be seen to be equivalent to that of the following inequality:

$$\frac{1}{t} - \left(\frac{4}{3} - \frac{4t}{9}\right) = \frac{(3-2t)^2}{9t} \geq 0$$

for t ε [1,2]. The inequality is actually true when $t > 0$, and false when $t < 0$.

(b) From the order preservation properties of definite integrals, the truth of inequality (*) on the closed interval [1,3] implies that

$$\ln(2) = \int_1^2 \frac{1}{t}\,dt \geq \int_1^2 \frac{4}{3} - \frac{4t}{9}\,dt$$

$$= \left(\frac{4t}{3} - \frac{2}{9}t^2\right)\Big|_1^2$$

$$= \left(\frac{8}{3} - \frac{8}{9}\right) - \left(\frac{4}{3} - \frac{2}{9}\right) = \frac{2}{3}.$$

To see that the inequality is actually *sharp*, it suffices to mumble that the two functions

$$f(t) = \frac{1}{t} \quad \text{and} \quad g(t) = \frac{4}{3} - \frac{4t}{9}$$

are continuous on [1,2], with $f(t) > g(t)$ except at $t = 3/2$. Thus the two integrals above are not equal.//