READ ME FIRST: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page. Eschew obfuscation.

1. (4 pts.) Find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so, find its limit.

$$\left(1 + \frac{1}{1}\right)^3,\ \left(1 + \frac{1}{2}\right)^5,\ \left(1 + \frac{1}{3}\right)^7,\ \left(1 + \frac{1}{4}\right)^9,\ \left(1 + \frac{1}{5}\right)^{11},\ \ldots .$$

$$a_n = \left(1 + \frac{1}{n}\right)^{2n+1} \quad for\ n \geq 1.$$

$$Thus,\ \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\left(1 + \frac{1}{n}\right)^n\right]^2 \times \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = e^2 \times 1 = e^2.$$

2. (4 pts.) Express the repeating decimal as a fraction, more specifically as a quotient of positive integers. [The fraction does not have to be in lowest terms.]

$$0.121121\underline{121}\ldots = \frac{121}{999}$$

either by using the "high school" method or summing an appropriate geometric series.

3. (4 pts) Using complete sentences and appropriate notation, give the precise ε - N definition of

(*) $\lim_{n\to\infty} a_n = L.$

// We write (*) above if L is a number such that, for each $\varepsilon > 0$, there is a positive integer N, dependent on ε, such that for every positive integer n, if $n \geq N$, then $|a_n - L| < \varepsilon$. //

4. (8 pts.) Complete the following by supplying the summand and filling in the blanks appropriately.

(a) A p-series is a series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

This series diverges if ____ $p \leq 1$ ____ and this series converges if ____ $p > 1$ ____ .

(b) A geometric series is a series of the form

$$\sum_{k=0}^{\infty} ar^k$$

This series diverges if ____ $|r| \geq 1$ ____ and this series converges if ____ $|r| < 1$ ____.

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5. (4 pts.) Give the precise mathematical definition of the sum of an infinite series,

(**) $$\sum_{k=1}^{\infty} a_k$$

// A number s is the sum of the series (**) above if

$$s = \lim_{n \to \infty} \sum_{k=1}^{n} a_k.$$

If the limit fails to exist, the series is said to diverge. //

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6. (8 pts.) Determine whether the series converges, and if so, find its sum.

(*a*) $$\sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^{k+2}$$

This is obviously a geometric series with r = -3/2 and a = -27/8. Since $|r| > 1$, the series diverges and there isn't a sum.

(*b*) $$\sum_{k=1}^{\infty} \left[\frac{1}{\ln(k+2)} - \frac{1}{\ln(k+3)}\right] = \lim_{n \to \infty} \sum_{k=1}^{n} \left[\frac{1}{\ln(k+2)} - \frac{1}{\ln(k+3)}\right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{\ln(3)} - \frac{1}{\ln(n+3)}\right] = \frac{1}{\ln(3)}$$

using the definition of the sum of an infinite series.

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7. (4 pts.) Find all values of x for which the series converges, and find the sum of the series for those values of x.

$$x^2 + \frac{x^3}{5} + \frac{x^4}{25} + \frac{x^5}{125} + \frac{x^6}{625} + \ldots$$

Evidently, this is a geometric series. Writing the series using sigma notation makes things easy. Thus,

$$\sum_{k=0}^{\infty} \frac{x^{k+2}}{5^k} = \sum_{k=0}^{\infty} x^2 \left(\frac{x}{5}\right)^k = \frac{x^2}{1 - \left(\frac{x}{5}\right)} = \frac{5x^2}{5 - x}$$

provided that $\left|\frac{x}{5}\right| < 1$, *or* $|x| < 5$, *or* $-5 < x < 5$.

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8. (4 pts.) Apply the divergence test and state what it tells you about each of the following series.

(*a*) $\sum_{k=1}^{\infty} \frac{k}{k+1}$ Since $\lim_{k \to \infty} \frac{k}{k+1} = 1 \neq 0$, divergence test implies that (a) diverges.

(*b*) $\sum_{k=1}^{\infty} \frac{\ln(k)}{\sqrt{k}}$ Since $\lim_{k \to \infty} \frac{\ln(k)}{\sqrt{k}} = 0$, divergence test provides no information concerning the convergence of (b).

9. (6 pts.) Let $f(x) = \sin(\pi x)$.
Obtain the 3rd Taylor polynomial of $f(x)$ about $x_0 = 1$.

$$p_3(x) = f(1) + \frac{f^{(1)}(1)}{1!}(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= -\frac{\pi}{1!}(x-1) + \frac{\pi^3}{3!}(x-1)^3$$

10. (4 pts.) Use root test to determine whether the series converges. If the test is inconclusive, say so.

$\sum_{k=1}^{\infty}\left(\frac{1}{2}+\frac{2}{k}\right)^k$ Since $\rho = \lim_{k\to\infty}\left[\left(\frac{1}{2}+\frac{2}{k}\right)^k\right]^{1/k} = \lim_{k\to\infty}\left(\frac{1}{2}+\frac{2}{k}\right) = \frac{1}{2} < 1$, root test implies that this series converges.

11. (6 pts.) Classify each of the following series as absolutely convergent (AC), conditionally convergent (CC), divergent (D), or none of the preceding, (N). Circle the letters corresponding to your choice. (No explicit proof is needed.)

(a) $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\sqrt{k}}{k^{1/4}+1}$ ~~(AC)~~ ~~(CC)~~ **(D)** ~~(N)~~

(b) $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k)!}$ **(AC)** ~~(CC)~~ ~~(D)~~ ~~(N)~~

(c) $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\sqrt{k}+4}$ ~~(AC)~~ **(CC)** ~~(D)~~ ~~(N)~~.

12. (4 pts.) Confirm that the integral test is applicable, and then use it to determine whether the following series converges:

$\sum_{k=1}^{\infty}\frac{1}{k^2+1}$ Let $f(x) = 1/(x^2+1)$ for $x \ge 1$. Plainly f is a positive continuous function, and $1/(k^2 + 1) = f(k)$ for $k \ge 1$. Since $f'(x) = (-2x)\times(1+x^2)^{-2} < 0$ for $x > 1$, it follows that f is decreasing for $x \ge 1$. This means that we may use f, defined above, in integral test to determine whether the given series converges. Since

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\left(\tan^{-1}(b) - \tan^{-1}(1)\right) = \frac{\pi}{4},$$

it follows from integral test that the given series of #12 converges.

13. (4 pts.) Consider $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}\,10^k}(x-1)^k$. From ratio test for absolute convergence, since $\lim_{k\to\infty}\left|\frac{u_{k-1}}{u_k}\right| = \frac{1}{10}|x-1|$. ,the radius of convergence is R = 10

Substitution of x = -9 yields $\sum_{k=1}^{\infty}\frac{(-1)^{2k}}{\sqrt{k}} = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$, and substitution of x = 11 yields $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$ Consequently, the interval of convergence is I = (-9,11].

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14. (4 pts.) Use ratio test to determine whether the series converges. If the test is inconclusive, say so.

$\sum_{k=1}^{\infty}\frac{5}{k^2}$ Since $\rho = \lim_{k\to\infty}\frac{5/(k+1)^2}{5/k^2} = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2 = 1$, ratio test is inconclusive. [Series converges, but ratio test ain't talkin'.]

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15. (4 pts.) Use comparison test to show the following series diverges.

$\sum_{k=1}^{\infty}\frac{\sqrt{k}}{k+4}$ First,

$$1/(5\sqrt{k}) \le \sqrt{k}/(k+4)$$

for $k \ge 1$. Since the series $\sum_{k=1}^{\infty}(1/(5k^{1/2}))$ diverges, being a nonzero multiple of a divergent p-series, comparison test implies that the series of problem #15 diverges.

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16. (4 pts.)

$$\ln(1.1) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k10^k}$$

Use the error estimate from alternating series test to find an upper bound on the absolute error when the finite sum

$$s_5 = \sum_{k=1}^{5}\frac{(-1)^{k+1}}{k10^k}$$

is used to approximate ln(1.1).

From the error estimate from the alternating series test, we have

$$|s_5 - \ln(1.1)| < \frac{1}{(6)10^6} .$$

Thus, M = 1/6000000 is an upper bound on the absolute error in the approximation.

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17. (4 pts.) Complete the following appropriately.

(a) If f has derivatives of all orders at x_0, then the Taylor series for f at $x = x_0$ is defined to be

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

(b) Suppose the function f can be differentiated five times on the interval I containing $x_0 = 2$ and that $|f^{(5)}(x)| \le 20$ for all x in I. Then, for all x in I,

$$|R_4(x)| \le \frac{20}{5!}|x-2|^5 .$$

18. (4 pts.) Determine whether the following sequence is eventually increasing or eventually decreasing or neither:

$\left\{ \frac{n!}{10^n} \right\}_{n=1}^{\infty}$ Since $\frac{a_{n+1}}{a_n} = \frac{n+1}{10} > 1 \text{ when } n \geq 10$, the sequence is eventually increasing.

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19. (8 pts.) (a) Suppose $f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$ and $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 0$.

What is the domain of the power series function *f* ? //Applying the ratio test for absolute convergence, since $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| |x-x_0| = 0$, the power series function *f* converges absolutely for each real number *x*. The domain of *f* is ℝ = (-∞,∞).//

(b) Suppose $g(x) = \sum_{k=0}^{\infty} b_k (x-x_0)^k$ and $\lim_{k\to\infty} \frac{|b_{k+1}|}{|b_k|} = +\infty$. What is the domain of the power series function *g* ? //Applying the ratio test for absolute convergence,since $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| |x-x_0| = \infty \text{ when } x \neq x_0$, it follows that the power series function *g* converges only at its center. The domain of *g* is { x_0 }.//

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20. (4 pts.) Let the sequence $\{a_n\}$ be defined recursively by

$$a_1 = 1, \text{ and } a_{n+1} = \frac{1}{2}\left(a_n + \frac{25}{a_n}\right)$$

for $n \geq 1$. Assuming the sequence converges, find its limit *L*.// First

$$L = \lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{25}{\lim_{n\to\infty} a_n}\right) = \frac{1}{2}\left(L + \frac{25}{L}\right)$$

Then routine algebra implies that $L^2 - 25 = 0$, so that $L = 5$ or $L = -5$. Since the sequence is nonnegative, the limit, if it exists, must also be nonnegative. As a consequence, $L = 5$.

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21. (4 pts.) Find a positive integer *N* so that if $n \geq N$, then

(*) $$\left|\frac{5n}{n+2} - 5\right| < \left(\frac{1}{2}\right)10^{-3}$$

and prove it provides the desired error bound.
// By doing a little routine algebra, it is easy to see that when $n \geq 1$, inequality (*) above is equivalent to

$$\left|\frac{-10}{n+2}\right| < \left(\frac{1}{2}\right)10^{-3} \Leftrightarrow \frac{10}{n+2} < \left(\frac{1}{2}\right)10^{-3} \Leftrightarrow 2\cdot 10^4 < n+2 \Leftrightarrow 19998 < n.$$

It follows that if we let $N = 19999$, then if $n \geq N$, then $n > 19998$, which is equivalent to (*). Just trace the double-headed arrow path backwards.//

Silly 10 Point Bonus: *[Found in c2-t3-b.pdf ??]*