READ ME FIRST: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page. Eschew obfuscation.

20. (4 pts.) Let the sequence $\{a_n\}$ be defined recursively by

$$a_1 = 1, \ \ and \ \ a_{n+1} = \frac{1}{2}\left(a_n + \frac{25}{a_n}\right)$$

for $n \geq 1$. Assuming the sequence converges, find its limit L.

First

$$L = \lim_{n \to \infty} a_{n+1} = \frac{1}{2}\left(\lim_{n \to \infty} a_n + \frac{25}{\lim_{n \to \infty} a_n}\right) = \frac{1}{2}\left(L + \frac{25}{L}\right)$$

Then the magic of routine algebra implies that $L^2 - 25 = 0$, so that $L = 5$ or $L = -5$. Since the sequence is nonnegative, the limit, if it exists, must also be nonnegative. As a consequence, $L = 5$.

Silly 10 Point Bonus: Prove that the sequence in Problem 20 is bounded below by $m = 0$ and eventually decreasing. A key piece is the minimum of

$$g(x) = \frac{1}{2}\left(x + \frac{25}{x}\right) \ for \ x > 0.$$

What does this now imply about the sequence??

First, this problem is a simple variant of Problem 28 of Section 9.2 of the ninth edition of Anton et al. You did that one didn't you?? You may, in fact, replace '25' above with any number A > 0 and '5' by the positive square root of A and have essentially the same problem and solution. Funny that ... .

Let

$$g(x) = \frac{1}{2}\left(x + \frac{25}{x}\right) \ for \ x > 0.$$

Then

$$g'(x) = \frac{1}{2}\left(1 - \frac{25}{x^2}\right) = \frac{(x-5)(x+5)}{2x^2} \ \ for \ x > 0.$$

It follows that $g'(x) < 0$ when $x < 5$, and $g'(x) > 0$ when $x > 5$. Evidently, then, g has an absolute minimum that occurs only at $x = 5$ and the minimum value is $g(5) = 5$. What this means is this:

(1) $$\frac{1}{2}\left(x + \frac{25}{x}\right) \geq 5$$

$$whenever \ x > 0, \ \ with \ equality \ only \ at \ x = 5.$$

Now let us return to the recursively defined sequence of Problem 20.

We shall first establish that the sequence $\{ a_n \}$ is a positive term sequence. A simple induction argument establishing this can be seen to hang on two observations: (a) From the recursive definition of the sequence, $a_1 = 1$, and (b) for each positive integer n, if $a_n > 0$, the recursive definition of a_{n+1} in terms of a_n implies that $a_{n+1} > 0$ is true. Thus, zero is a lower bound for the sequence.

Next, using that the sequence is positive-termed, the recursive definition, and inequality (1) on the preceding page, one can see that we have

$$a_n > 5 \text{ whenever } n \geq 2. \tag{2}$$

We shall find that (2) above is valuable in proving that the sequence is eventually decreasing. [A proof of this is another elementary induction argument. You don't have to do it.]

We shall show the sequence eventually decreasing in two different ways: (A) We shall study the difference $a_{n+1} - a_n$, and (B) we shall study the quotient a_{n+1}/a_n.

(A) We study the difference $a_{n+1} - a_n$:

From the recursive definition of the sequence,

$$a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{25}{a_n}\right) - a_n = \frac{(5 - a_n)(5 + a_n)}{2a_n}$$

for $n \geq 1$ by doing elementary algebra. Inequality (2) above implies that this difference is negative when $n \geq 2$. Thus, the sequence is eventually decreasing.

(B) We study the quotient a_{n+1}/a_n:

From the recursive definition of the sequence,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2}\left(a_n + \frac{25}{a_n}\right)}{a_n} = \frac{1}{2}\left(1 + \frac{25}{a_n^2}\right) \tag{3}$$

for $n \geq 1$ by doing elementary algebra. When $n \geq 2$, inequality (2) above implies that

$$25 < a_n^2 \text{ so that } \frac{25}{a_n^2} < 1.$$

Consequently, the quotient in (3) above will be less than 1 when $n \geq 2$. Thus, the sequence is eventually decreasing.

What does this now imply about the sequence?? Since the sequence is bounded below by zero and is eventually decreasing, it must have a limit as $n \to \infty$.//