STUDENT NUMBER: 0000000 EXAM NUMBER: 00

Read Me First:
Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Remember that what is illegible or incomprehensible is worthless. Communicate. Eschew obfuscation. Good Luck! [Total Points Possible: 120]

BONUS Part (a): State the Mean-Value Theorem for Integrals and use it to prove the second part of the Fundamental Theorem of Calculus.

Mean-Value Theorem of Integrals. If f is a continuous function defined on a closed interval $[a,b]$, then there is a number x^* in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(x^*)(b-a) .$$

2nd Part of the Fundamental Theorem. Let $f(x)$ be a function that is continuous on a non-degenerate interval I, and suppose that a in any point in I. If the function g is defined on I by the formula

$$g(x) = \int_a^x f(t)\,dt,$$

for each x in I, then $g'(x) = f(x)$ for each x in I.

Remark. You may, of course, reproduce the argument presented by Anton, et al, in Section 5.6 of the 9^{th} Edition or Section 6.6 of the 8^{th} Edition that avoids revealing any of the neat ε - antics that can be done. We, however, will revel in that silliness.

Proof with ε - Antics.

First observe that the function g is well-defined due to the continuity of the function f on the interval I and properties of the definite integral.

We next obtain an expression in terms of a definite integral for the usual difference quotient used in computing a derivative from its definition . Let x ε I, Then

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \\ &= \frac{1}{h}\int_x^{x+h} f(t)\,dt \\ &\textit{provided } h \neq 0 \textit{ and } x+h \in \mathrm{I}.\end{aligned}$$

Now we shall restrict our attention to the where x is either a left endpoint or an interior point of I. We shall show that in this case that

$$\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = f(x) .$$

To this end, let $\varepsilon > 0$ be arbitrary. Since f is continuous from the right at x, there is a small number $\delta > 0$ so that if t satisfies $0 \le t - x < \delta$, then $|f(t) - f(x)| < \varepsilon$. Grab this wee δ and suppose that h is a number that satisfies $0 < h < \delta$. If $0 < h < \delta$, then $[x,x+h] \subset [x,x+\delta)$. Applying the Mean-Value Theorem for Integrals to f on the interval $[x,x+h]$, it follows that there is a number t^* in $[x,x+h] \subset [x,x+\delta)$ where

$$\int_x^{x+h} f(t)\ dt = f(t^*)((x+h) - x) \quad ,$$

so

$$\frac{1}{h}\int_x^{x+h} f(t)\ dt = f(t^*) \quad .$$

Thus,

$$\left| \frac{g(x+h) - g(x)}{h} - f(x) \right| = \left| \frac{1}{h}\int_x^{x+h} f(t)\ dt - f(x) \right|$$

$$= |f(t^*) - f(x)| < \epsilon.$$

This completes the proof that the right-handed derivative of g at x is $f(x)$ whenever x is either a left endpoint that is in the interval or is an interior point of the interval. [Yes, we verified the ε - δ definition.]

The proof that

$$\lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h} = f(x)$$

when x is either a right endpoint or an interior point of I is similar. To this end, let $\varepsilon > 0$ be arbitrary. Since f is continuous from the left at x, there is a small number $\delta > 0$ so that if t satisfies $-\delta < t - x \le 0$, then $|f(t) - f(x)| < \varepsilon$. Grab this wee δ and suppose that h is a number that satisfies $-\delta < h < 0$. If $-\delta < h < 0$, then we also have $[x+h,x] \subset (x-\delta,x]$. Applying the Mean-Value Theorem for Integrals to f on the interval $[x+h,x]$, it follows that there is a number t^* in the interval $[x+h,x] \subset (x-\delta,x]$ where

$$\int_{x+h}^{x} f(t)\ dt = f(t^*)(x - (x+h)) \quad ,$$

so

$$\frac{1}{-h}\int_{x+h}^{x} f(t)\ dt = f(t^*) \quad .$$

Thus,

$$\left| \frac{g(x+h) - g(x)}{h} - f(x) \right| = \left| \frac{1}{h}\int_x^{x+h} f(t)\ dt - f(x) \right|$$

$$= \left| \frac{1}{(-h)}\int_{x+h}^{x} f(t)\ dt - f(x) \right|$$

$$= |f(t^*) - f(x)| < \epsilon.$$

This completes the proof that the left-handed derivative of g at x is $f(x)$ whenever x is either a right endpoint that is in the interval or is an interior point of the interval. [Yes, we satisfied the ε - δ definition.]

Putting the pieces together, now, we are finished. //

11. (10 pts.) Suppose

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\,8^k}(x-3)^k$$

(a) By using sigma notation and term-by-term differentiation *as done in class*, obtain a power series for $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k8^k}(x-3)^k\right] = \sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{(-1)^{k+1}}{k8^k}(x-3)^k\right]$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k8^k}\frac{d}{dx}\left[(x-3)^k\right] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{8^k}(x-3)^{k-1}$$

(b) By using sigma notation and integrating term-by-term *as done in class*, obtain an infinite series whose sum has the same numerical value as that of the following definite integral. [We are working with the power series f above.]

$$\int_3^5 f(x)\,dx = \int_3^5 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k8^k}(x-3)^k\,dx = \sum_{k=1}^{\infty} \int_3^5 \frac{(-1)^{k+1}}{k8^k}(x-3)^k\,dx$$

$$= \sum_{k+1}^{\infty} \frac{(-1)^{k+1}}{k8^k}\int_3^5 (x-3)^k\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}2^{k+1}}{k(k+1)8^k}$$

BONUS Part (b): Identify the function f given by the power series in Problem 11 above.

Remark. This is similar to an old bonus found online in the Test Tombs in the Calculus II materials in Bonus Noise: c2-t3-bo.pdf in Fall, 2005. Like that problem, this may be solved in more than one way.

Solution 1. Doing straightforward algebra, observe that

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\,8^k}(x-3)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{x-3}{8}\right)^k$$

$$= \ln\left(1+\left(\frac{x-3}{8}\right)\right) = \ln\left(\frac{x+5}{8}\right)$$

if $-1 < (x - 3)/8 \le 1$ or $-5 < x \le 11$. You need only to have the solution to 3 (e) above,

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = \ln(1+x) \quad \textit{for } x \in (-1,1] ,$$

in your bio-computer for pattern matching.//

Solution 2. As an alternative approach, you might realize that you can obtain f' in a closed form since the solution to Problem 11 (a) is a geometric series.

Observe that

$$f'(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k8^k}\frac{d}{dx}[(x-3)^k] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{8^k}(x-3)^{k-1}$$

$$= \sum_{j=0}^{\infty} \frac{1}{8}\left(\frac{(-1)(x-3)}{8}\right)^j = \frac{1}{5+x} \quad when \ |x-3| < 8.$$

Clearly $f(3) = 0$. Thus, the function f satisfies the wee initial value problem, $f'(x) = 1/(x + 6)$ for x ε (-5,11) with $f(3) = 0$. Using the 2nd Part of The Fundamental Theorem of Calculus, we have

$$f(x) = \int_3^x \frac{1}{t+5}\,dt = \ln(x+5) - \ln(8) = \ln\left(\frac{x+5}{8}\right)$$

for x ε (-5,11).// (End of "hard" solution).