Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "$\Rightarrow$" denotes "implies" , and "$\Leftrightarrow$" denotes "is equivalent to". Vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page.

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1. (10 pts.) (a) Obtain parametric equations for the line tangent to the graph of the helix defined by

$$\mathbf{r}(t) = < \cos(t), \sin(t), t/\pi >$$

when $t = \pi$. // Evidently, we need a point on the line, the point of contact with the helix given by the terminal point of the position vector $\mathbf{r}(\pi) = < -1, 0, 1 >$. We also require a vector giving the direction. Plainly, the *unit tangent vector*, $\mathbf{T}(\pi)$, will do the job, but the simpler to compute $\mathbf{r}'(\pi)$ has all the direction information needed, too. Thus, since

$$\mathbf{r}'(t) = < -\sin(t), \cos(t), 1/\pi > ,$$

$\mathbf{r}'(\pi) = < 0, -1, 1/\pi>$. Consequently, a vector equation for the line tangent to the graph of the helix is

$$< x, y, z > = \mathbf{r}(\pi) + t\cdot\mathbf{r}'(\pi) = < -1, 0, 1 > + t\cdot< 0, -1, 1/\pi>.$$

Some parametric equations: $x = -1$, $y = -t$, and $z = 1 + \pi^{-1}t$.

(b) Does the tangent line at $t = \pi$ in part (a) intersect the plane defined by $x + y + z = 0$? If "yes", where??

A normal vector for the plane is $\mathbf{n} = < 1, 1, 1 >$. Since we have $\mathbf{r}'(\pi)\cdot\mathbf{n} = 0 + (-1) + (1/\pi) = (1/\pi) - 1 \neq 0$, the tangent line and the plane are not parallel. They intersect when t satisfies the equation $-1 + (-t) + (1 + (1/\pi)t) = 0$. Solving this equation reveals that $t = 0$. Thus, the point of intersection is $(x,y,z) = (-1,0,1)$. [Don't assume 't = 0' is typical.]

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2. (10 pts.) (a) Find the unit vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and the curvature $\kappa(t)$ for the helix defined by

$$\mathbf{r}(t) = < \cos(t), t, \sin(t)>.$$

First, $\mathbf{r}'(t) = < -\sin(t), 1, \cos(t) >$, so that $|\mathbf{r}'(t)| = 2^{1/2}$. Consequently,

$$\mathbf{T}(t) = 2^{-1/2}< -\sin(t), 1, \cos(t) >,$$

$\mathbf{T}'(t) = 2^{-1/2}< -\cos(t), 0, -\sin(t) >$ with $|\mathbf{T}'(t)| = 2^{-1/2}$, and $\mathbf{N}(t) = < -\cos(t), 0, -\sin(t) >$. Finally, $\kappa(t) = |\mathbf{T}'(t)| / |\mathbf{r}'(t)| = 1/2$. [Sorry about the bold kappa, but the word processor ... . You must know that curvature is a number.]

(b) Locate the center $C(x_0,y_0,z_0)$ of the circle of curvature when $t = \pi/2$.// The coordinates of the center $C(x_0,y_0,z_0)$ of the circle of curvature when $t = \pi/2$ satisfy the vector equation

$$\begin{aligned} <x_0,y_0,z_0> &= \mathbf{r}(\pi/2) + \rho(\pi/2)\mathbf{N}(\pi/2) \\ &= <0, \pi/2, 1> + 2\cdot<0 ,0 , -1> = < 0 , \pi/2, -1 >. \end{aligned}$$

3. (10 pts.) (a) Obtain an equation for the plane containing the point (3,2,1) and parallel to the plane defined by the equation 10x - 25y + 13z = 75.

Planes are parallel precisely when their normal vectors are parallel. Consequently, a point-normal equation for the desired plane is

$$10(x - 3) - 25(y - 2) + 13(z - 1) = 0.$$

In standard form we have something like 10x - 25y + 13z = -7, but you really don't have to go this far.

(b) Obtain parametric equations for the line containing the point (3,2,1) and parallel to the line defined by the symmetric equations

$$\frac{x-7}{-4} = \frac{y+23}{-11} = \frac{z-\pi}{8\pi}$$

Two lines in 3 - space are parallel when their direction vectors are parallel. A direction vector that will do is **d** = < -4, -11, 8π >. Thus a set of parametric equations for the line we want is x = 3 - 4t, y = 2 - 11t, and z = 1 + 8πt.

4. (10 pts.) (a) Obtain an equation for the plane that contains the three points (1,-1,1), (0,2,0), and (0,0,3).

Label the points A = (1,-1,1), B = (0,2,0), and C = (0,0,3). Then two vectors in the plane are

$$\mathbf{v} = \vec{AB} = < -1, 3, -1 > \text{ and } \mathbf{w} = \vec{AC} = < -1, 1, 2 >.$$

These vectors clearly are not parallel. A normal vector for the plane containing the three points is

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = < 7, -(-3), 2 > = < 7, 3, 2 >.$$

A point-normal equation for the plane is

$$7(x - 1) + 3(y + 1) + 2(z - 1) = 0.$$

An equation in standard form is 7x + 3y + 2z = 6. [Not required.]

(b) What is the area of the triangle in 3-space with vertices at (1,-1,1), (0,2,0), and (0,0,3). [Hint: You may already have done most of the work.]

The area of the triangle is

$$\text{Area}(\Delta) = (1/2)|\mathbf{n}| = (1/2)|\mathbf{v} \times \mathbf{w}| = (1/2)(62)^{1/2}.$$

5. (10 pts.) Obtain an arc-length parameterization for the curve **r**(t) = < t, cos(t), sin(t) > in terms of the initial point (π, -1, 0). Rather than overloading the symbol **r**, write this new parameterization as **R**(s). How are **R** and **r** related?

First, since **r**(π) = <π, -1, 0>, the (signed) distance from the point given by **r**(π) to **r**(t) is

$$\begin{aligned} s = \varphi(t) &= \int_{\pi}^{t} |\mathbf{r}'(u)|\, du \\ &= \int_{\pi}^{t} |\langle 1, -\sin(u), \cos(u)\rangle|\, du\ . \\ &= \int_{\pi}^{t} 2^{1/2}\, du = 2^{1/2}(t - \pi) \end{aligned}$$

Solving for t in terms of s yields

$$t = 2^{-1/2}s + \pi \quad so\ that\ \varphi^{-1}(t) = 2^{-1/2}t + \pi\ .$$

Thus,

$$\mathbf{R}(s) = \mathbf{r}(\varphi^{-1}(s)) = \mathbf{r}(2^{-1/2}s+\pi) = \langle 2^{-1/2}s+\pi,\ \cos(2^{-1/2}s+\pi),\ \sin(2^{-1/2}s+\pi)\rangle$$
.

Evidently, **R**(s) = **r**(φ^{-1}(s)), or equivalently, **r**(t) = **R**(φ(t)).

6. (10 pts.) (a) Find the center and radius of the sphere with the following equation:

$$x^2 + y^2 + z^2 + 4x - 6y + 8z = 0$$

Completing the square thrice yields

$$(x + 2)^2 + (y + 3)^2 + (z + 4)^2 = 29.$$

Thus, the center is (-2,3,-4) and the radius is r = $(29)^{1/2}$.

(b) Obtain an equation for the sphere that has center at the point (1,-1,1) and that is tangent to the line in 3-space that is defined by the vector equation < x, y, z > = t< 1, 2, 3>.

The radius, of course, is the distance from the point to the line. [I can think of three ways to obtain this. Can you?] One way to obtain this is as follows: The radius is the shortest length of the vector function **v**(t) = <t - 1, 2t + 1, 3t - 1> built using <1,-1,1> as the initial point and taking as the terminal point an arbitrary point on the line. The distance is shortest for t_0 satisfying **v**(t_0)·<1,2,3> = 0. [Why?] Computing this dot product and solving for t_0 in the little linear equation that results yields t_0 = 1/7. Thus,

$$r = |\langle -6/7,\ 9/7\ ,\ -4/7 \rangle| = (1/7)(133)^{1/2}.$$

An equation for the sphere:

$$(x - 1)^2 + (y + 1)^2 + (z - 1)^2 = 133/49.$$

7. (10 pts.) Write **c** in the form r**a** + s**b** where r and s are real numbers when **a** = <2,2>, **b** = <-2,2>, and **c** = <-3,2>. That is, write **c** as a linear combination of **a** and **b** if possible.

From the definition of linear combination or **c** being written in the form r**a** + s**b** where r and s are real numbers, we can write c in the desired form if, and only if r and s are real numbers so that the vector equation

$$\mathbf{c} = r\mathbf{a} + s\mathbf{b}$$

is true.
Now from the definitions of vector equality and the vector operations, this is equivalent to r and s being solutions to the linear system

$$-3 = 2r - 2s$$

$$2 = 2r + 2s.$$

Solving this system yields the equivalent system

$$r = -1/4$$

$$s = 5/4.$$

Thus, **c** = (-1/4)**a** + (5/4)**b**. Of course, you may also write this as

$$\langle -3,2\rangle = (-1/4)\langle 2,2\rangle + (5/4)\langle -2,2\rangle.$$

8. (10 pts.) (a) Find the direction angles of the vector represented by $\vec{PQ}$ when the points P(1,-1,0) and Q(3,4,5) are given.

Since $\vec{PQ} = \langle 2,5,5\rangle$ *and thus*, $|\vec{PQ}| = \sqrt{54}$, we have

$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{54}}\right), \quad \beta = \cos^{-1}\left(\frac{5}{\sqrt{54}}\right), \text{ and } \quad \gamma = \cos^{-1}\left(\frac{5}{\sqrt{54}}\right).$$

(b) What point $R(x_0,y_0,z_0)$ is 2/3 of the way from P to Q??

The coordinates of the point $R(x_0,y_0,z_0)$ satisfy the vector equation

$$\begin{aligned}\langle x_0, y_0, z_0 \rangle &= \langle 1,-1,0\rangle + \frac{2}{3}\vec{PQ}\\ &= \langle 1,-1,0\rangle + \frac{2}{3}\langle 2,5,5\rangle\\ &= \langle 3/3,-3/3,0\rangle + \langle 4/3,10/3,10/3\rangle\\ &= \langle 7/3,7/3,10/3\rangle.\end{aligned}$$

So the point is R(7/3,7/3,10/3).

9. (10 pts.) Suppose that an object is moving in a fixed plane with its acceleration given by **a**(t) = -32**j**. Suppose that the initial position of the object is **r**(0) = <1,0> and the initial velocity of the object is **v**(0) = < 2, 4>.

(a) Find the velocity of the object, **v**(t), as a function of time.// It follows easily from the vector-valued version of the Fundamental Theorem of Calculus that

$$\mathbf{v}(t) = \int_0^t \boldsymbol{a}(u)\,du + \langle 2,4\rangle$$
$$= \int_0^t \langle 0,-32\rangle\,du + \langle 2,4\rangle = \langle 0,-32t\rangle + \langle 2,4\rangle = \langle 2,4-32t\rangle.$$

(b) Find the position of the object, **r**(t), as a function of time.// Similarly, it follows easily from the vector-valued version of the Fundamental Theorem of Calculus that

$$\mathbf{r}(t) = \int_0^t \boldsymbol{v}(u)\,du + \langle 1,0\rangle$$
$$= \int_0^t \langle 2,4-32u\rangle\,du + \langle 1,0\rangle = \langle 2t,4t-16t^2\rangle + \langle 1,0\rangle$$
$$= \langle 2t+1,\ 4t-16t^2\rangle.$$

(c) Obtain an equation for the parabola that is the path of the object.// An equivalent set of parametric equations for **r**(t) is $x = 2t + 1$ and $y = 4t - 16t^2$. Solving for t in the "x" equation and substituting the result into the equation for "y" provides us with $t = (1/2)(x - 1)$ and $y = 2(x-1) - 4(x-1)^2$, an equation for the parabola traversed. [This may be expanded...]

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10. (10 pts.) (a) Find the exact value of the acute angle θ of intersection of the two planes defined by the two equations

$$x - 2y = -5 \quad \text{and} \quad y - 4z = 20.$$

$\theta = \cos^{-1}(\,2/(85)^{1/2}\,)$. [Remember ... acutely?]

(b) Obtain a set of parametric equations for the line of intersection of the two planes in part (a).

$\begin{cases} x-2y = -5 \\ \quad y = 20+4z \end{cases}$ is equivalent to $\begin{cases} x = 35+8z \\ y = 20+4z \end{cases}$. A set of

parametric equations: $\begin{cases} x = 35+8t \\ y = 20+4t \\ z = t \end{cases}$.

Silly 10 Point Bonus: Jeopardy!! The strange parameterization of the unit circle minus the point (-1,0) given by

$$\vec{r}(t) = \left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right\rangle \ \text{for } -\infty < t < \infty$$

may be obtained using *two* quite different ideas. Reveal this magic in detail. Hint: Substitution deviousness & a different thought of lines do the jobs. Say where your work is here: