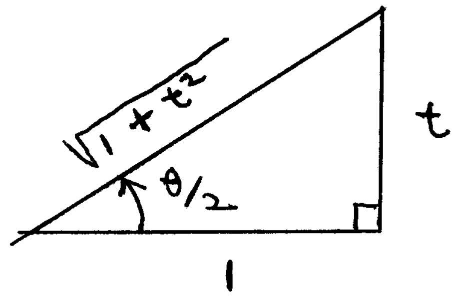
Silly 10 Point Bonus: Jeopardy!! The strange parameterization of the unit circle minus the point (-1,0) given by

$$(*) \qquad \mathbf{r}(t) = <\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}> \textit{ for } -\infty < t < \infty$$

may be obtained using *two* quite different ideas. Reveal this magic in detail. Hint: Substitution deviousness & a different thought of lines do the jobs. Say where your work is here:
My work follows.



First, we deal with the devious substitution magic. It turns out that the classical "z-substitution" that allows antiderivatives of rational functions of sine and cosine to be transformed into rational functions does the trick. Simply set $t = \tan(\theta/2)$ for $-\pi < \theta < \pi$. By considering the triangle above, which represents the equation $t = \tan(\theta/2)$ when $\theta/2$ is acute,you may readily read off that

$$\sin(\theta/2) = t/(1 + t^2)^{1/2},$$

and

$$\cos(\theta/2) = 1/(1 + t^2)^{1/2}.$$

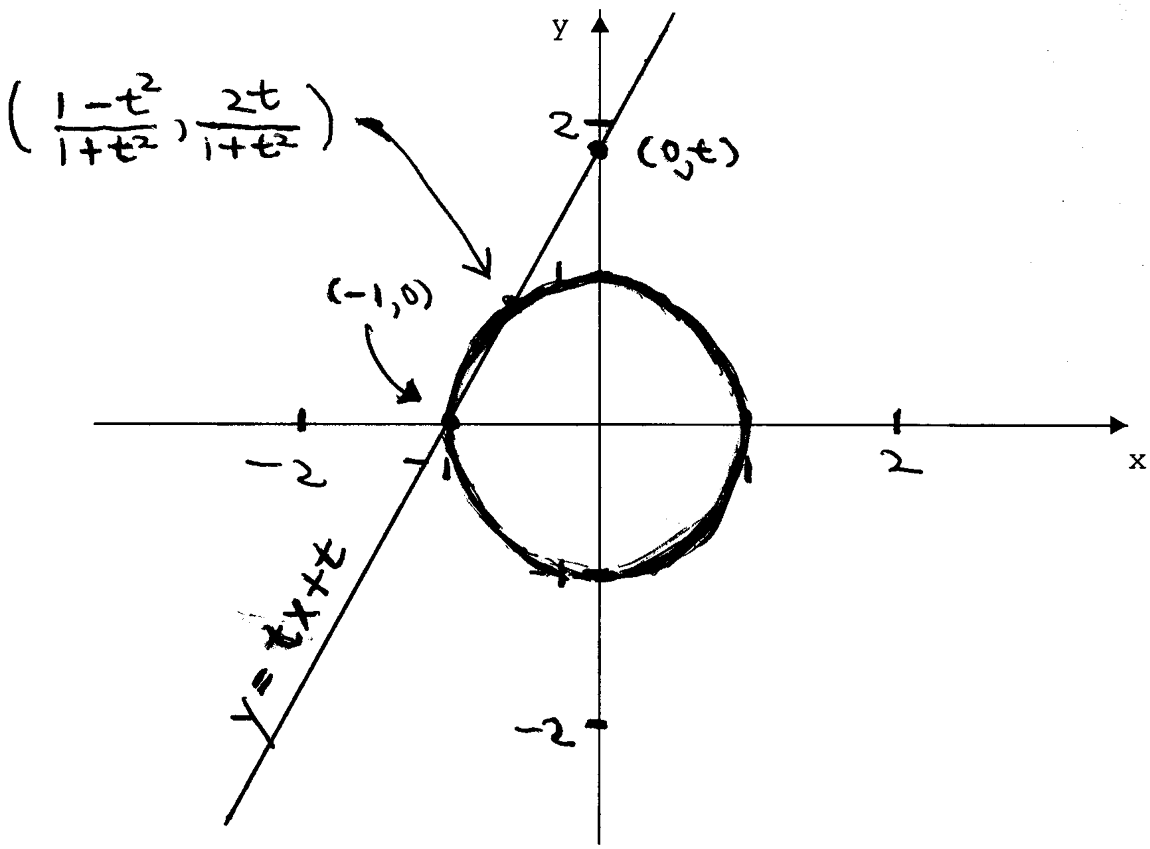
The oddness of sine and evenness of cosine will allow you then to deal with θ satisfying $-\pi < \theta < 0$. Note that we have theta given in terms of t via the equation $\theta = 2\cdot\tan^{-1}(t)$, too.
Thus,

$$\sin(\theta) = \sin(2(\theta/2)) = 2\cdot\sin(\theta/2)\cos(\theta/2) = 2t/(1 + t^2),$$

and

$$\cos(\theta) = \cos(2(\theta/2)) = \cos^2(\theta/2) - \sin^2(\theta/2) = (1 - t^2)/(1 + t^2).$$

Evidently, the point ($\cos(\theta)$, $\sin(\theta)$) lies on the unit circle. By using the right sides of the two equations immediately above one can build (*). The equation $\theta = 2\cdot\tan^{-1}(t)$ and the two equations immediately above allow one to understand more regarding the parameterization given by (*) above.



A second way to obtain (*) above is in some way reminiscent of how one maps the complex plane into the Riemann sphere. [Look above.] For each real number t, build the straight line through the points (-1, 0) on the unit circle and the point (0,t) on the y-axis. An equation for this line is

(**) $$y = tx + t.$$

One can determine where this line intersects the unit circle, defined by the equation

(***) $$x^2 + y^2 = 1,$$

by finding the solutions to the system consisting of these two equations. Replacing y in the equation for the circle leads to the quadratic equation $(t^2+1)x^2 + 2t^2x + (t^2-1) = 0$. Using the quadratic formula provides two solutions to the quadratic equation: $x = -1$ or $x = (1 - t^2)/(1 + t^2)$. The linear equation (**) now allows us to obtain the corresponding y values. It follows that the solution to the system consisting of (**) and (***) is the pair of ordered pairs (-1,0) and the pair used to define the parameterization (*),

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$