Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page.

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1. (10 pts.) (a) If the cylindrical coordinates for a point P are given by $(r, \theta, z) = (3, (7/6)\pi, -1)$, then the rectangular coordinates of the point are given by

$$(x, y, z) = (3\cdot\cos((7/6)\pi), 3\cdot\sin((7/6)\pi), -1)$$

$$= (-(3/2)\cdot(3)^{1/2}, -3/2 , -1).$$

(b) If the spherical coordinates of a point for a point P are given by $(\rho, \phi, \theta) = (4, (1/6)\pi, (2/3)\pi)$, then the rectangular coordinates of the point are given by

$$(x, y, z) = (4\cdot\sin(\pi/6)\cos(2\pi/3),4\cdot\sin(\pi/6)\sin(2\pi/3),4\cdot\cos(\pi/6))$$

$$= (-1, 3^{1/2}, 2(3)^{1/2}).$$

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2. (10 pts.) (a) Suppose the rectangular coordinates for a point P are given by $(x, y, z) = (-2, -1, 2)$. Obtain cylindrical and spherical coordinates for this point. In doing this, ensure that $0 \leq \theta < 2\pi$.

Plainly, $r = 5^{1/2}$, $\rho = 3$, $\phi = \cos^{-1}(2/3)$, and $\theta = \tan^{-1}(1/2) + \pi$ since the terminal side of θ is in the third quadrant. Consequently, the cylindrical coordinates for the point are given by

$$(r, \theta, z) = (5^{1/2}, \tan^{-1}(1/2) + \pi, 2).$$

The spherical coordinates are given by

$$(\rho, \phi, \theta) = (3, \cos^{-1}(2/3), \tan^{-1}(1/2) + \pi).$$

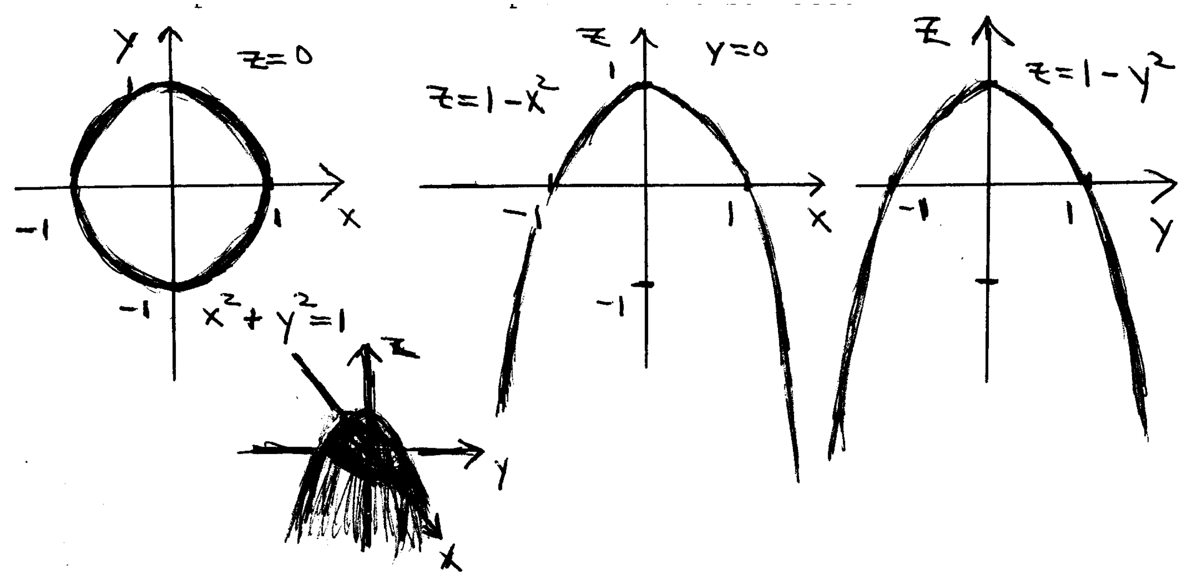
(b) Describe the graph of the equation $\rho = 4\cos(\phi)$ as precisely as possible. [Evidently this is in spherical coordinates.] You may wish to convert this equation to an equivalent rectangular equation to do this.

$$\rho = 4\cos(\phi) \quad\Leftrightarrow\quad \rho^2 = 4\rho\cos(\phi) \quad\Leftrightarrow\quad x^2 + y^2 + z^2 = 4z.$$

By transposing the z term on the right and completing the square, one can obtain easily $x^2 + y^2 + (z -2)^2 = 4$, an equation for a sphere centered at (0,0,2) with a radius of 2. Hello??

3. (10 pts.) (a) Write an equation for the surface generated when the curve defined by the equation $y = 4 - x^2$ in the x,y - plane is revolved around the y - axis.// The curve in the xy-plane defined by $y = 4 - x^2$ may be realized as the zero set for the function $h(x,y) = 4 - x^2 - y$, that is the set of pairs (x,y) where $h(x,y) = 0$. [This is a formality to deal easily with the issue of uniform substitution!] The surface obtained when the curve is revolved around the y-axis consists of triples (x_0,y_0,z_0) with the property that the intersection of the surface with the plane $y = y_0$ parallel to the xz-plane is a point on a circle with center $(0,y_0,0)$ and radius r_0 obtained from the point (r_0,y_0) lying on the curve defined by $h(x,y) = 0$. Since (x_0,y_0,z_0) lies on the circle, $r_0 = ((x_0)^2 + (z_0)^2)^{1/2}$. Substituting (r_0,y_0) into $h(x,y) = 0$ yields $0 = 4 - ((x_0)^2 + (z_0)^2) - y_0$. Dropping the subscripts and doing a little algebra gives us the desired equation: $y = 4 - x^2 - z^2$.

(b) Do the three 2-space sketches of the traces in each of the coordinate planes of the surface defined by $z = 1 - x^2 - y^2$. Work below and label carefully. Then on the back of page 1 attempt to do a 3 - space sketch in the plane of the surface.



4. (10 pts.) (a) What is the largest possible domain of the function $f(x,y) = \ln(x^2 + y^2 - 4)$

The domain of f is the set $\{ (x,y) : x^2 + y^2 - 4 > 0 \}$ which is the same as $\{ (x,y) : x^2 + y^2 > 4 \}$, the exterior of a disk centered at (0,0) with radius 2.

(b) Consider the function $f(x,y) = x^2 - y$. Obtain an equation for the level curve for this function through the point (1,-2) in the x,y - plane. [Hint: What is the *level* for the level curve?]

The *level* is simply the function value. In this case, that turns out to be $f(1,-2) = 3$. Thus, the level curve through the point (1,-2) is the set of ordered pairs satisfying $f(x,y) = 3$, or more precisely, the set of ordered pairs satisfying $x^2 - y = 3$.

5. (10 pts.) (a) Using complete sentences and appropriate notation, give the ε - δ definition for

$$(*) \qquad \lim_{(x,y) \to (a,b)} f(x,y) = L.$$

We say that the limit of f(x,y) is L as (x,y) approaches (a,b) and write (*) if L is a number such that, for each $\varepsilon > 0$, there is a number $\delta > 0$ with the following property: if (x,y) is any ordered pair in the domain of f with $0 < ((x-a)^2 + (y-b)^2)^{1/2} < \delta$, then it must follow that $|f(x,y) - L| < \varepsilon$.

(b) In the example where the authors were showing that

$$\lim_{(x,y) \to (a,b)} xy = ab,$$

they asserted that if they took f(x,y) = x and g(x,y) = y, then it followed from the definition of limit that

$$\text{(i)} \quad \lim_{(x,y) \to (a,b)} f(x,y) = a \quad \text{and} \quad \text{(ii)} \quad \lim_{(x,y) \to (a,b)} g(x,y) = b.$$

Choose one of equations (i) or (ii), indicate to me which you have chosen, and then prove the equation is true using the definition.// We shall show (ii). The proof of (i) is similar. Thus, let $\varepsilon > 0$ be arbitrary. Now choose any $\delta > 0$ with $\delta \le \varepsilon$. We shall now verify that this δ does the dastardly deed. To this end, let (x,y) be any pair of real numbers with $0 < ((x-a)^2 + (y-b)^2)^{1/2} < \delta$. It then follows that

$$\begin{aligned} |g(x,y) - b| &= |y - b| = ((y-b)^2)^{1/2} \\ &\le ((x-a)^2 + (y-b)^2)^{1/2} < \delta \le \varepsilon.// \end{aligned}$$

6. (10 pts.) (a) If f(x,y) is a function of two variables, state the definition of the partial derivative of f with respect to y.// The partial derivative of f with respect to y is the function $f_y(x,y)$ defined by

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

wherever the limit exists.

(b) Let $f(x,y) = xy^2 - 2$. Using the definition, compute $f_y(x,y)$.

$$\begin{aligned} f_y(x,y) &= \lim_{k \to 0} \frac{f(x,y+k)-f(x,y)}{k} \\ &= \lim_{k \to 0} \frac{[x(y+k)^2-2]-[xy^2-2]}{k} \\ &= \lim_{k \to 0} \frac{[xy^2+2xyk+xk^2-2]-[xy^2-2]}{k} \\ &= \lim_{k \to 0} (2xy+kx) \\ &= 2xy. \end{aligned}$$

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7. (10 pts.) Find an equation for the plane tangent to the surface $z = f(x,y)$ at the point $P = (5, 2, 9)$ when

$$f(x,y) = x^2 - 4y^2.$$

Since $f_x(x,y) = 2x$ and $f_y(x,y) = -8y$, $f_x(5,2) = 10$ and $f_y(5,2) = -16$. An equation for the plane tangent to the surface is

$$z - 9 = 10(x - 5) - 16(y - 2).$$

There are, of course, infinitely many equations that are equivalent to this one.

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8. (10 pts.) (a) Evaluate the following limit.

$$\lim_{(x,y)\to(-1,2)} \frac{5 - x^2}{1 + xy} = -4$$

Observe that the function we are dealing with is a rational function that is not defined only where $xy = -1$. Thus we are dealing with a continuous function.

(b) Is f defined by $f(x,y) = \begin{cases} \dfrac{\tan(3(x^2+y^2))}{x^2+y^2}, & (x,y)\neq(0,0) \\ 2, & (x,y)=(0,0) \end{cases}$

continuous at (0,0)? A complete explanation is required. Details ... .

Since $f(0,0) = 2$ and

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\tan(3(x^2+y^2))}{x^2+y^2}$$
$$= \lim_{r\to 0^+} \frac{\tan(3r^2)}{r^2}$$
$$\overset{(L'H)}{=} \lim_{r\to 0^+} \frac{6r\,\sec^2(3r^2)}{2r} = 3$$

by using L'Hopital's rule after doing a conversion to polar coordinates, f is not continuous at (0,0). [You do not have to convert to polar coordinates, but replacing $x^2 + y^2$ seems needed. So some sort of substitution is probably desirable.]

9. (10 pts.) Find every point on the surface $z = f(x,y)$ where the tangent plane is horizontal when

$$f(x,y) = 3x^2 + 12x + 4y^3 - 6y^2.$$

Since f is a polynomial, f has continuous first order partial derivatives

$$f_x(x,y) = 6x + 12 = 6(x + 2),$$

and

$$f_y(x,y) = 12y^2 - 12y = 12y(y - 1).$$

f has horizontal tangent planes at each point (x,y) in the domain where $f_x(x,y) = 0$ and $f_y(x,y) = 0$. This is equivalent to $x = -2$ and $y = 0$, or $x = -2$ and $y = 1$. Thus, f has horizontal tangent planes at $(x,y) = (-2,0)$ or $(x,y) = (-2,1)$. The actual points of tangency are $(-2,0,-12)$ and $(-2,1,-14)$.

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10. (10 pts.) Reveal how one finds the maximum and minimum values of $f(x,y) = 2xy$ on the circular disk R, where $R = \{ (x,y) : x^2 + y^2 \le 4 \}$. Oh yes, do find the maximum and minimum values, too. [Hint: Is an arctic boundary cool?]

Since f is a continuous function defined on the closed and bounded set R, the Extreme Value Theorem guarantees that f has absolute extrema of both flavors. We need only examine the functional values of f at its critical points, if there are any, and on the boundary of the set R, the circle with radius 2 centered at the origin, $C = \{(x,y) : x^2 + y^2 = 4 \}$, which we will parameterize using sine and cosine, thus:

$$C = \{(x,y) : x = 2\cdot\cos(\theta),\ y = 2\cdot\sin(\theta),\ \theta \in [0,2\pi] \}.$$

[*This is not the only way to deal with the boundary!!!*]
Since $f_x(x,y) = 2y$ and $f_y(x,y) = 2x$ in the interior of R, the only critical point f has is (0,0) and $f(0,0) = 0$. To study f on the boundary, set $g(\theta) = f(2\cdot\cos(\theta),2\cdot\sin(\theta)) = 8\cdot\cos(\theta)\sin(\theta)$. Since $g(\theta) = 4\cdot\sin(2\theta)$ for $\theta \in [0,2\pi]$, we can read off the extreme values of f. The maximum value of f on R is 4 and occurs at $(2^{1/2},2^{1/2})$ and $(-2^{1/2},-2^{1/2})$. The minimum value of f on R is -4 and occurs at $(-2^{1/2},2^{1/2})$ and $(2^{1/2},-2^{1/2})$.

Silly 10 Point Bonus: Suppose that $f(x,y) = g(x)$, where g is the function defined on the real line by $g(x) = 1$ if $x \ne 0$ and $g(0) = 0$. (a) Using the definition, show

$$\lim_{x \to 0} g(x) = 1 .$$

(b) Then prove that if y_0 is any real number, f(x,y) does not have a limit as $(x,y) \longrightarrow (0,y_0)$. [This doesn’t require use of the definition!] (c) Jeopardy: To which *natural* proposition does the pair of functions f and g provide a counter-example? Say where your work is here: