Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Generic vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page neatly.

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1. (10 pts.)
Let $f(x,y) = (x^2 + y^2)^{1/2}$. Compute the gradient of f at (4,-3), and then use it to compute $D_{\mathbf{u}}f(4,-3)$, where **u** is the unit vector in the same direction as **v** = <-12,5>.

$$\nabla f(x,y) = \left\langle \frac{1}{2}(x^2+y^2)^{-1/2}(2x),\ \frac{1}{2}(x^2+y^2)^{-1/2}(2y) \right\rangle$$

$$= \left\langle \frac{x}{(x^2+y^2)^{1/2}},\ \frac{y}{(x^2+y^2)^{1/2}} \right\rangle$$

$$\nabla f(4,-3) = \left\langle \frac{4}{5},\ -\frac{3}{5} \right\rangle$$

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \left\langle -\frac{12}{13},\ \frac{5}{13} \right\rangle$$

$$D_{\mathbf{u}}f(4,-3) = \nabla f(4,-3)\cdot\mathbf{u} = \left\langle \frac{4}{5},\ -\frac{3}{5} \right\rangle \cdot \left\langle -\frac{12}{13},\ \frac{5}{13} \right\rangle = -\frac{63}{65}.$$

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2. (10 pts.)
Let $f(x,y) = e^{-3x + 4y}$ and let $P_0 = (0,0)$.
(a) Compute the rate of change of f(x,y) at P_0 in the direction in which f(x,y) increases most rapidly.

Since

$$\nabla f(x,y) = \left\langle -3e^{-3x+4y},\ 4e^{-3x+4y} \right\rangle ,$$

it follows that $\nabla f(0,0) = \langle -3, 4\rangle$. Thus, the rate we want is

$$|\nabla f(0,0)| = |\langle -3, 4\rangle| = 5.$$

(b) Find a unit vector in the direction in which f(x,y) increases most rapidly at P_0.

The unit vector we want is the one giving the direction of the vector $\nabla f(0,0) = \langle -3, 4\rangle$. . Thus we want the unit vector

$$\mathbf{u} = \left\langle -\frac{3}{5},\ \frac{4}{5} \right\rangle .$$

3. (10 pts.) Make the gradient do a perp walk.

(a) When $f(x,y) = x^4 + xy + y^2$, compute $\nabla f(2,-3)$. Label your expressions correctly. Don't scratch.

Since $\nabla f(x,y) = < 4x^3 + y,\ x + 2y >$, $\nabla f(2,-3) = < 29, -4 >$.

(b) Use the result of (a) to produce an equation for the line tangent to the graph of $x^4 + xy + y^2 = 19$ at the point $(x_0,y_0) = (2,-3)$.

Since $\nabla f(2,-3)$ is perpendicular to the graph of

$$x^4 + xy + y^2 = 19$$

at (2,-3), if (x,y) lies on the line tangent to the graph at (2,-3), then $< 29, -4 > \cdot < x - 2, y - (-3) > = 0$.
Of course, by doing a little routine algebra, you can transform this last equation into $y = (29/4)x - (70/4)$.

(c) Use the result of (a) to produce an equation for the line perpendicular to the graph of $x^4 + xy + y^2 = 19$ and passing through the point $(x_0,y_0) = (2,-3)$. [Hint: There is no restriction on the *kind* of equation.]

In this case, the gradient provides the direction for the line. The simplest equation is a vector equation that you can easily transform into the usual slope-intercept varmint. Of course if you understand the plane perpendicularity game you could get the slope by using the result of part (b), above.

Vector equation: $< x, y > = < 2, -3 > + t\cdot\nabla f(2,-3)$

$$= < 2, -3 > + t\cdot< 29, -4 >$$

Almost a Varmint: $y = (-4/29)(x - 2) - 3$

4. (10 pts.) (a) Find the differential, dw, of

$$w = (x^{1/2} + y^{1/2})^2.$$

$$dw = 2(x^{1/2} + y^{1/2})(1/2)x^{-1/2}\,dx + 2(x^{1/2} + y^{1/2})(1/2)y^{-1/2}\,dy$$

$$= (1 + (y/x)^{1/2})dx + ((x/y)^{1/2} + 1)dy$$

(b) Reveal, in detail, how to use differentials to approximate

$$((17)^{1/2} + (24)^{1/2})^2.$$

Let $f(x,y) = (x^{1/2} + y^{1/2})^2$. We want to approximate f(17,24). Set $(x_0,y_0) = (16,25)$ and $(x_0 + \Delta x, y_0 + \Delta y) = (17,24)$. Then $\Delta x = 1$, $\Delta y = -1$, and

$$f(17,24) = f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

$$= 81 + (9/4)(1) + (9/5)(-1) = \ldots = 81.45.$$

5. (10 pts.) Locate and classify the critical points of the function $f(x,y) = x^3 + 3xy^2 - 3x$. Use the second partials test in making your classification. (Fill in the table below after you locate all the critical points. // Since $f_x(x,y) = 3x^2 + 3y^2 - 3$ and $f_y(x,y) = 6xy$, the critical points of f are given by the solutions to the following system:

$$x^2 + y^2 = 1 \text{ and } xy = 0.$$

Carefully dealing with the logical operators provides us with four critical points: (0,1), (0,-1), (1,0), and (-1,0).

Crit.Pt.	f_{xx} @ c.p.	f_{yy} @ c.p.	f_{xy} @ c.p.	Δ @ c.p.	Conclusion
(x,y)	6x	6x	6y	$36x^2-36y^2$	
(0,1)	0	0	6	-36	Saddle Point
(0,-1)	0	0	-6	-36	Saddle Point
(1,0)	6	6	0	36	Local Minimum
(-1,0)	-6	-6	0	36	Local Maximum

6. (10 pts.) (a) Compute $\partial z/\partial x$ and $\partial z/\partial y$ as functions of x, y, and z assuming that $z = f(x,y)$ when $x^2 + y^2 + z^2 = 9$. [These are implicit differentiations, of course.]

By pretending z is a function of the two independent variables x and y and performing partial differentiations on both sides of the equation above, we have

$$2x+2z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z},$$

and

$$2y+2z\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

(b) Using your results from part (a), obtain an equation for the plane tangent to the surface defined by $x^2 + y^2 + z^2 = 9$ at the point P(1,2,2).

Since

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2,2)} = -\frac{1}{2},$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2,2)} = -\frac{2}{2},$$

an equation for the plane is $z - 2 = (-1/2)(x - 1) - (y - 2)$.

7. (10 pts.) By using Lagrange multipliers, find the extreme values of $f(x,y) = y^2 - x^2$ and precisely where they occur on the circle defined by the equation $x^2 + y^2 = 4$.

Set $g(x,y) = x^2 + y^2 - 4$. Then (x,y) is on the circle defined by $x^2 + y^2 = 4$ precisely when (x,y) satisfies $g(x,y) = 0$. Since $\nabla g(x,y) = \langle 2x, 2y\rangle$, $\nabla g(x,y) \neq \langle 0, 0\rangle$ when (x,y) is on the circle given by $g(x,y) = 0$. Plainly f and g are smooth enough to satisfy the hypotheses of the Lagrange Multiplier Theorem. Thus, if a constrained local extremum occurs at (x,y), there is a number λ so that $\nabla f(x,y) = \lambda\nabla g(x,y)$. Now

$$\nabla f(x,y) = \lambda\nabla g(x,y) \Rightarrow \langle -2x , 2y\rangle = \lambda\langle 2x,2y\rangle \Rightarrow 0 = 4xy$$

$$\Rightarrow x = 0 \text{ or } y = 0$$

by performing a little routine algebraic magic. Solving each of the systems consisting of (a) $x^2 + y^2 = 4$ and $x = 0$, and (b) $x^2 + y^2 = 4$ and $y = 0$, yields the desired critical points, (0,2), (0,-2), (2,0), and (-2,0).
It turns out $f(0,2) = f(0,-2) = 4$ and $f(2,0) = f(-2,0) = -4$. So -4 is the minimum and occurs at (2,0) and (-2,0), and 4 is the maximum and occurs at the two points (0,2) and (0,-2).

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8. (10 pts.) Chain, chain, chain, chain of ... .
(a) Suppose that $w = \ln(2x + 3y)$, $x = r\cdot\cos(\theta)$ and $y = r\cdot\sin(\theta)$. Compute $\partial w/\partial\theta$.

$$\begin{aligned}\partial w/\partial\theta &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial\theta}\\ &= \frac{2}{2x+3y}(-r)\sin(\theta) + \frac{3}{2x+3y}(r)\cos(\theta)\\ &= \frac{(-2r)\sin(\theta) + (3r)\cos(\theta)}{2r\cos(\theta) + 3r\sin(\theta)}.\end{aligned}$$

(b) Suppose that $x = h(y,z)$ satisfies the equation $F(x,y,z) = 0$, and that $F_x \neq 0$. Show how to compute $\partial x/\partial z$ in terms of the partial derivatives of F.// We'll write this twice, just for the fun of it. First, using classical curly d notation, we have

$$0 = \frac{\partial F(x,y,z)}{\partial z} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial z} \Rightarrow 0 = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial z}$$
$$\Rightarrow \frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x}.$$

If we use subscript notation for the partial derivatives, we set $g(y,z) = F(h(y,z), y, z)$. Then

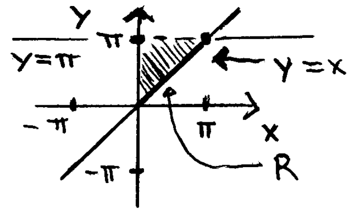
$$\begin{aligned}0 &= g_z(y,z)\\ &= F_x(h(y,z), y, z)h_z(y, z) + F_z(h(y,z), y, z),\end{aligned}$$

which implies that

$$\partial x/\partial z = h_z(y,z) = - F_z(h(y,z), y, z)/F_x(h(y,z), y, z).$$

9. (10 pts.) It turns out that $h(y) = \sin(y)/y$ does not have an elementary antiderivative. Despite that you can evaluate the following iterated integral. Do this by reversing the order of integration and then evaluating the new iterated integral you obtain. [Hint: It helps to sketch the region of integration, R.]

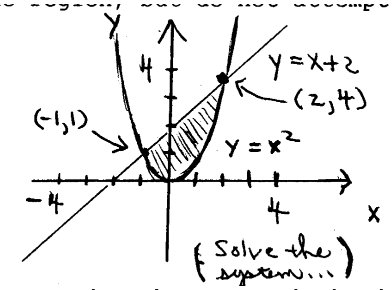
$$\begin{aligned}\int_0^\pi\int_x^\pi \frac{\sin(y)}{y}\,dy\,dx &= \int_0^\pi\int_0^y \frac{\sin(y)}{y}\,dx\,dy\\ &= \int_0^\pi \frac{\sin(y)}{y}\left[\int_0^y 1\,dx\right]dy\\ &= \int_0^\pi \frac{y\sin(y)}{y}dy\\ &= \int_0^\pi \sin(y)\,dy\\ &= -\cos(\pi) - (-\cos(0)) = 2.\end{aligned}$$



Note: Although the integral appears to be improper, it really isn't. Why??

10. (10 pts.) (a) Let R be the region bounded by the curves defined by $y = x^2$ and $y = x + 2$. Write an iterated double integral that gives the area of the region, but do not attempt to evaluate the iterated integral.

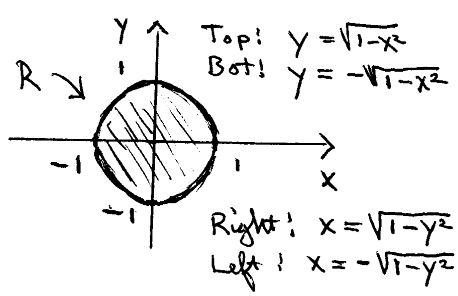
$$\begin{aligned}area(R) &= \iint_R 1\,dA\\ &= \int_{-1}^2\int_{x^2}^{x+2} 1\,dy\,dx\end{aligned}$$



(b) Set up, but do not attempt to evaluate the iterated double integral that will give the numerical value for the volume of the solid bounded by the cylinders in 3-space defined by $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$. [Hint: You should only need to sketch the xy-cylindrical stuff to obtain the limits of integration.] *//*

Here the top surface is $z = (1 - y^2)^{1/2}$ and the bottom surface is $z = -(1 - y^2)^{1/2}$.

$$\begin{aligned}V &= \iint_R (1-y^2)^{1/2} - (-(1-y^2)^{1/2})\,dA\\ &= \int_{-1}^1\int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} 2(1-y^2)^{1/2}\,dy\,dx\\ &= \int_{-1}^1\int_{-(1-y^2)^{1/2}}^{(1-y^2)^{1/2}} 2(1-y^2)^{1/2}\,dx\,dy\end{aligned}$$



Silly 10 Point Bonus: Suppose $f(x,y)$ is differentiable at an interior point (x_0,y_0) in its domain. Pretend there are at least three distinct unit vectors **u** satisfying the following equation: $D_uf(x_0,y_0) = 0$. Does it follow as a consequence that this equation must be true for all unit vectors? Proof?? Where???