---

**Read Me First:** Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Generic vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page neatly.

---

**Silly 10 Point Bonus:** Suppose f(x,y) is differentiable at an interior point $(x_0,y_0)$ in its domain. Pretend there are at least three distinct unit vectors **u** satisfying the following equation: $D_{\mathbf{u}}f(x_0,y_0) = 0$. Does it follow as a consequence that this equation must be true for all unit vectors? Proof?? Where???

The key thing here is to guess that the existence of at least three distinct unit vectors **u** in the plane satisfying the equation $D_{\mathbf{u}}f(x_0,y_0) = 0$ implies that $\nabla f(x_0,y_0) = \langle 0, 0\rangle$, and thus that $D_{\mathbf{u}}f(x_0,y_0) = 0$ for all unit vectors. Having done that, one must set out to prove that guess.

I’ll show you a couple of arguments. The first is indirect and quite brief. The second is longer and uses trigonometry in an inessential way to deal with linear algebraic matters.

Before we get down to serious work, observe that the equation $D_{\mathbf{u}}f(x_0,y_0) = 0$ is equivalent to

(*) $$f_x(x_0,y_0)\cdot u_1 + f_x(x_0,y_0)\cdot u_2 = 0,$$

where the components of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ satisfy the equation

(**) $$1 = \|\mathbf{u}\|^2 = (u_1)^2 + (u_2)^2.$$

**The Brief Argument:** If $\nabla f(x_0,y_0) \neq \langle 0, 0\rangle$, then (*) above is the equation of a non-degenerate line through the origin (0,0). There are exactly two unit vectors **u** satisfying (*) since the line defined by (*) intersects the circle defined by (**) exactly twice. Thus, if the gradient at $(x_0,y_0)$ is nonzero in the plane, there are exactly two unit vectors **u** satisfying (*). If there are at least three unit vectors satisfying (*), then it is not the case that exactly two unit vectors satisfy (*). Hence it is not the case that $\nabla f(x_0,y_0) \neq \langle 0, 0\rangle$. So ... .//

[Your pet logician might mutter something resembling "modus tollens" and wince here. A mathematician might mumble about the contrapositive.]

**The Not So Brief Argument:** Since each unit vector in the plane may be written as

$$\mathbf{u} = < \cos(\theta), \sin(\theta) >$$

for a unique $\theta$ with $0 \le \theta < 2\pi$, and we have assumed that there are at least three distinct unit vectors with $D_{\mathbf{u}}f(x_0,y_0) = 0$, there must be two numbers $\theta_1$ and $\theta_2$ in the interval $[0,2\pi)$ with

$$f_x(x_0,y_0)\cdot\cos(\theta_1) + f_x(x_0,y_0)\cdot\sin(\theta_1) = 0,$$

and

$$f_x(x_0,y_0)\cdot\cos(\theta_2) + f_x(x_0,y_0)\cdot\sin(\theta_2) = 0,$$

and $\theta_2 - \theta_1$ different from any integer multiple of $\pi$. [Why? At most two of the vectors may be parallel.] Now the neat thing is that if we view the system of equations immediately above as a linear system with the unknowns consisting of $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$, the determinant of the coefficient matrix is

$$\cos(\theta_1)\sin(\theta_2) - \cos(\theta_2)\sin(\theta_1) = \sin(\theta_2 - \theta_1) \neq 0.$$

It follows that the solution to the homogeneous linear system must be $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$, the trivial solution. Consequently, the gradient of f at $(x_0,y_0)$ must be the zero vector and the directional derivative there must be zero in every direction. So $D_{\mathbf{u}}f(x_0,y_0) = 0$ for all unit vectors if there are at least three distinct unit vectors satisfying the equation.

**Remarks:** (1) This bit of trivia is not true if f has three or more variables. Why?

(2) You could deal with the linear system that arises without the halloween magic, the trig or treat stuff. I just thought it neat how one can use it here to deal with the determinant of the coefficient matrix.

(3) The first, indirect argument could be made via the equation $D_{\mathbf{u}}f(x_0,y_0) = |\nabla f(x_0,y_0)|\cdot\cos(\theta)$, where $\theta$ is the angle between $\nabla f(x_0,y_0)$ and the unit vector **u**. Note that you must assume $\nabla f(x_0,y_0) \neq < 0, 0>$ in order to deduce anything at all about $\theta$.