Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page.

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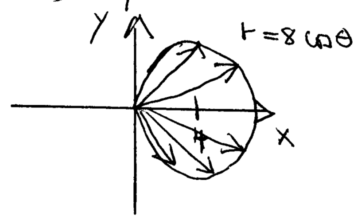
1. (10 pts.) Set up, but do not attempt to evaluate the iterated double integral in polar coordinates that gives the area of the region in the plane that is inside the circle defined by the equation $r = 8\cdot\cos(\theta)$. Label your expressions appropriately.

$$r^2 = 8r\cos\theta \Leftrightarrow x^2+y^2=8x$$
$$\Leftrightarrow x^2-8x+16+y^2=16$$
$$\Leftrightarrow (x-4)^2+y^2=16$$



$$Area(R) = \int_{-\pi/2}^{\pi/2}\int_0^{8\cos(\theta)} r\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{8\cos(\theta)} r\,dr\,d\theta.$$

Remember that we must keep $r \geq 0$, Folks??? Try drawing the rectangular r,θ auxiliary graph first to see how the circle gets traced. See Section 9.2.

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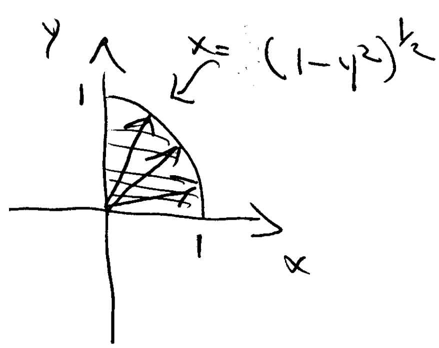
2. (10 pts.) Convert the given iterated integral into an iterated integral in polar coordinates that has the same numerical value and is easier to evaluate, perhaps. Do not attempt to evaluate the polar integral.

$$\int_0^1\int_0^{(1-y^2)^{1/2}} \sin(x^2+y^2)\,dx\,dy = \iint_R \sin(x^2+y^2)\,dA = \int_0^{\pi/2}\int_o^1 \sin(r^2)\,r\,dr\,d\theta.$$

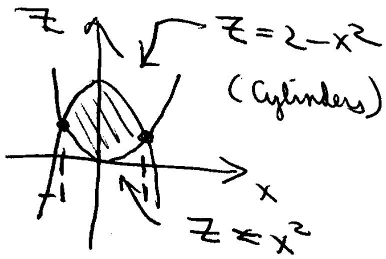
Of course it helps to observe that we have $0 \leq x \leq (1 - y^2)^{1/2}$ when $0 \leq y \leq 1$. Thus, the region, R, may be realized as follows:

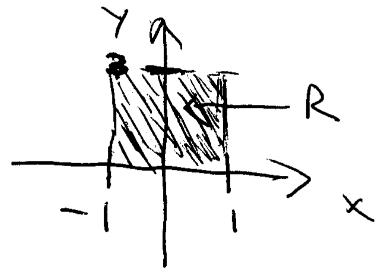


3. (10 pts.) Write down the triple iterated integral in cartesian coordinates that would be used to evaluate

$$\iiint_T f(x,y,z)\ dV\ ,$$

where $f(x,y,z) = x + y$ and T is the region between the surfaces $z = 2 - x^2$ and $z = x^2$ with $0 \le y \le 3$, but do not attempt to evaluate the triple iterated integral you have obtained. [Sketching the traces in the coordinate planes might help.]



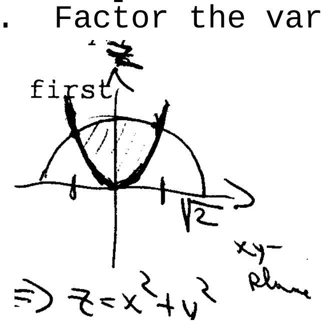


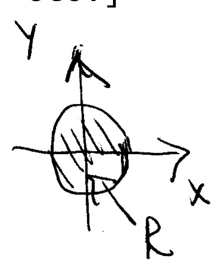
$$\begin{aligned}\iiint_T f(x,y,z)\ dV &= \iint_R \left[\int_{x^2}^{2-x^2} x+y\ dz\right] dA\\ &= \int_0^3\int_{-1}^1\int_{x^2}^{2-x^2} x+y\ dz\,dx\,dy,\quad or\\ &= \int_{-1}^1\int_0^3\int_{x^2}^{2-x^2} x+y\ dz\,dy\,dx.\end{aligned}$$

4. (10 pts.) Write down the triple iterated integral in cylindrical coordinates that provides the numerical value of the volume of the region in 3-space bounded above by the spherical surface $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$, but do not attempt to evaluate the integral you obtain. [Converting the equations for the surfaces to cylindrical first cuts down on the clutter??]

$$\iiint_T 1\ dV \ = \ \int_0^{2\pi}\int_0^1\int_{r^2}^{(2-r^2)^{1/2}} r\,dz\,dr\,d\theta.$$

The the equations for the two surfaces in cylindrical coordinates are $r^2 + z^2 = 2$ and $z = r^2$. The sphere with radius $2^{1/2}$ intersects the paraboloid when $r = 1$. The projection of the solid onto the xy-plane is a disk. [Just solve the system. You end up looking at $r^4 + r^2 - 2 = 0$. Factor the varmint, etc.]

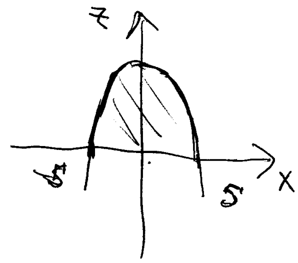




5. (10 pts.) Compute the surface area of the part of paraboloid defined by $z = 25 - x^2 - y^2$ that lies above the xy-plane. [Of course *above* means $z \geq 0$.]

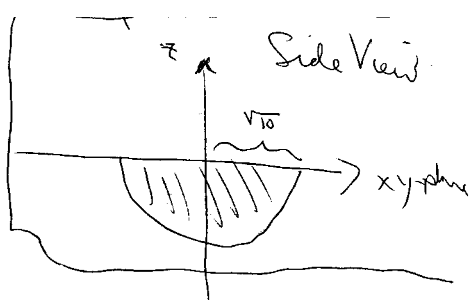
By using the definition, converting to polar coordinates at an opportune time, and then using the u-substitution $u = 4r^2 + 1$, we have

$$\begin{aligned} SA &= \iint_R \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 \right)^{1/2} dA \\ &= \iint_R ((-2x)^2+(-2y)^2+1)^{1/2}\, dA \\ &= \iint_R (4(x^2+y^2)+1)^{1/2}\, dA \\ &= \int_0^{2\pi}\int_0^5 (4r^2+1)^{1/2}\, r\,dr\,d\theta \\ &= 2\pi\int_0^5 (4r^2+1)^{1/2} r\,dr \\ &= \frac{\pi}{4}\int_1^{101} u^{1/2}\, du \\ &= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{101} = \frac{\pi}{6}((101)^{3/2}-1). \end{aligned}$$



6. (10 pts.) Write down, but do not attempt to evaluate the triple iterated integral in spherical coordinates that provides the volume of the solid T that is bounded above by the xy-plane and below by the sphere defined by $x^2 + y^2 + z^2 = 10$.

$$\iiint_T 1\, dV = \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{(10)^{1/2}} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$



7. (10 pts.) (a) If **F**(x,y) = < xy, x+y > and C is the curve defined y = x^2 with -1 ≤ x ≤ 1, evaluate the following line integral. [Obtain **r**(t) first. Equivalent integrals may simplify your life --- or not.]

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} \mathbf{F}(t,t^2)\cdot \mathbf{r}'(t)dt$$
$$= \int_{-1}^{1} < t^3,\ t + t^2 > \cdot < 1,\ 2t > dt$$
$$= \int_{-1}^{1} 3t^3 + 2t^2\,dt$$
$$= \int_{-1}^{1} 2t^2\,dt = \ldots = \frac{4}{3}.$$

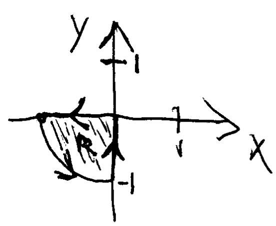
after setting **r**(t) = < t, t^2 > for -1 ≤ t ≤ 1.

(b) Starting at the point (-1,0), a particle traverses the lower semi-circle of $x^2 + y^2 = 1$ until it reaches the y-axis. Then it moves vertically until it reaches the origin. Finally it returns to its starting point by moving along the x-axis. Use Green's Theorem to compute the work done on the particle by the force field defined by **F**(x,y) = < -2y, 2x > for (x,y) in $\mathbb{R}^2$. [Draw a picture. This is easy??]

The work done can be obtained from

$$W = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(-2y)\,dA = 4\iint_R 1\,dA = 4\left(\frac{\pi}{4}\right) = \pi,$$

since R is one-fourth of the unit circle pie:



8. (10 pts.) Let **F**(x,y,z) = <x^2, -2, xyz>. Compute the divergence and the curl of the vector field **F**.

(a) $$div\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(-2)+\frac{\partial}{\partial z}(xyz) = 2x + xy.$$

(b) $$curl\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2 & xyz \end{vmatrix}$$
$$= < xz, -(yz), 0 >$$

9. (10 pts.) (a) Show that the vector field

$$F(x,y) = < 2xe^y, x^2e^y >$$

is actually a gradient field by producing a function $\phi(x,y)$ such that $\nabla\phi(x,y) = \mathbf{F}(x,y)$ for all (x,y) in the plane.

Evidently $\frac{\partial}{\partial y}(2xe^y) = 2xe^y = \frac{\partial}{\partial x}(x^2e^y)$ *for each* $(x,y) \in \mathbb{R}^2$. Then

$$\frac{\partial\phi}{\partial x}(x,y) = 2xe^y \Rightarrow \phi(x,y) = \int 2xe^y\,dx = x^2e^y + c(y),$$

Therefore,

$$x^2e^y = \frac{\partial\phi}{\partial y}(x,y) = \frac{\partial}{\partial y}(x^2e^y) + \frac{dc}{dy}(y) \quad\Rightarrow\quad \frac{dc}{dy}(y) = 0 \quad\Rightarrow\quad c(y) = c_0,$$

for some number c_0. Hence, $\phi(x,y) = x^2e^y + c_0$.

(b) Using the Fundamental Theorem of Line Integrals, evaluate the following integral.

By using part (a) of this problem,

$$\int_{(0,0)}^{(-1,1)} 2xe^y dx + x^2e^y dy = (x^2e^y)\Big|_{(0,0)}^{(-1,1)} = (-1)^2e^1 - (0)^2e^0 = e.$$

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10. (10 pts.) Use the substitution u = xy and v = y/x to compute the area of the region R *in the first quadrant* bounded by the lines defined by y = x and y = 2x, and the hyperbolas defined by xy = 1 and xy = 2. Thus, evaluate the following integral by doing the suggested substitution. // Keep in mind the substitution defines T^{-1}.

$$\iint_R 1\,dx\,dy = \iint_S 1\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{u,v}, \textit{where } S = [1,2]\times[1,2] \textit{ in the uv-plane},$$
$$= \int_1^2\int_1^2 \frac{1}{2v}dv\,du = \frac{1}{2}\int_1^2 \ln(2)du = \ln(\sqrt{2}).$$

The u,v bounding curves turn out to be simple constants: u = 1, u = 2, v = 1, and v = 2. Although the x,y region is painful to draw, the inverse image of R, S, is not. Observe that x > 0 and y > 0 is equivalent to u > 0 and v > 0. You may then solve for x and y in terms of u and v to obtain $x = u^{1/2}v^{-1/2}$ and $y = u^{1/2}v^{1/2}$. [There are several ways to do this.] Computing $\partial(x,y)/\partial(u,v)$ then is routine. You could also have obtained $\partial(x,y)/\partial(u,v)$ by computing $\partial(u,v)/\partial(x,y)$ and using the relation, $\partial(x,y)/\partial(u,v)\partial(u,v)/\partial(x,y) = 1$. It turns out that it is easy to see that $\partial(u,v)/\partial(x,y) = 2y/x$ here. So $\partial(x,y)/\partial(u,v) = 1/2v$.

Silly 10 Point Bonus: What's the numerical value of the following double integral, where R is the first quadrant.

$$\iint_R e^{-(x^2+y^2)}dA$$

[Where? You don't have room here to reveal the needed details.]