Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Vector objects must be denoted by using arrows. Since a correct answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page clearly.

1. (10 pts.) Obtain an equation for the plane that is tangent to the sphere with equation

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 51$$

at the point (2, 3, -4). [Hint: Who’s normal?]

Let P(1,2,3) and Q(2,3,-4) be the labelled center of the sphere and the point of tangency. Then a vector perpendicular to the plane is given by

$$\boldsymbol{n} = \vec{n} = \vec{PQ} = \langle 2 - 1, 3 - 2, -4 - 3 \rangle = \langle 1, 1, -7 \rangle .$$

Hence, an equation for the desired plane is given by

$$(x - 2) + (y - 3) - 7(z + 4) = 0.$$

An equivalent equation is

$$x + y - 7z = 33.$$

2. (10 pts.) Obtain parametric equations for the line obtained when the two planes defined by x + 3y + 6z = 10 and x - 2y + z = 5 intersect.

$$\begin{cases} x+3y+6z = 10 \\ x-2y+z = 5 \end{cases} \text{ is equivalent to } \begin{cases} x = 7-3z \\ y = 1-z. \end{cases}.$$ This was obtained by doing standard "row" or "equation" operations. A set of parametric equations:

$$\begin{cases} x = 7-3t \\ y = 1-t \\ z = t \ , \ t \in \mathbb{R}. \end{cases}.$$

Kindly note that there are several legitimate ways to reach a correct system of parametric equations, and there are infinitely many such systems. You can check by substitution into the equations for the planes. For the parametric equations above, we can see that

$$(7 - 3t) + 3(1 - t) + 6(t) = 10$$

and

$$(7 - 3t) - 2(1 - t) + (t) = 5$$

identically, for each $t \in \mathbb{R}$.

3. (10 pts.) (a) Obtain parametric equations for the line that contains the point (25, -24, 23) and the center of the sphere defined by the following equation:

$$x^2 + y^2 + z^2 + 4x - 6y + 8z = 0$$

Completing the square thrice yields

$$(x + 2)^2 + (y - 3)^2 + (z + 4)^2 = 29.$$

Thus, the center is (-2,3,-4). Using the center and the point (25, -24, 23), we can build a vector giving the direction for the line, namely **v** = < 27, -27, 27 >. Consequently, a set of parametric equations for the line is given by x = 25 + t, y = -24 - t, and z = 23 + t. [What happened to "27"??]

(b) What is the radius of the sphere that has its center at the point (1,2,3) and that is tangent to the line in 3-space that is defined by the vector equation < x, y, z > = t< 1, -1, 1>.

The radius, of course, is the distance from the point to the line. One way to obtain this is as follows: The radius is the shortest length of the vector function **v**(t) = <t-1, -t-2, t-3> built using <1,2,3> as the initial point and taking as the terminal point an arbitrary point on the line. The distance is shortest for t_0 satisfying $\mathbf{v}(t_0)\cdot$<1,-1,1> = 0. [Why?] Computing this dot product and solving for t_0 in the little linear equation that results yields $t_0 = 2/3$. Thus,

$$r = |< -1/3, -8/3 , -7/3 >| = (1/3)(114)^{1/2}.$$

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4. (10 pts.) Suppose that an object is moving in a fixed plane with its acceleration given by **a**(t) = <0,-32>. Suppose that the initial position of the object is **r**(0) = <0,1> and the initial velocity of the object is **v**(0) = < 2, 2>.

(a) Find the velocity of the object, **v**(t), as a function of time.// It follows easily from the vector-valued version of the Fundamental Theorem of Calculus that

$$\begin{aligned}\boldsymbol{v}(t) &= \int_0^t \boldsymbol{a}(u)\,du + \langle 2,2\rangle \\ &= \int_0^t \langle 0,-32\rangle\,du + \langle 2,2\rangle = \langle 0,-32t\rangle + \langle 2,2\rangle = \langle 2,2-32t\rangle.\end{aligned}$$

(b) Find the position of the object, **r**(t), as a function of time.// Similarly,

$$\begin{aligned}\boldsymbol{r}(t) &= \int_0^t \boldsymbol{v}(u)\,du + \langle 0,1\rangle \\ &= \int_0^t \langle 2,2-32u\rangle\,du + \langle 0,1\rangle = \langle 2t,2t-16t^2\rangle + \langle 0,1\rangle \\ &= \langle 2t,\ 1+2t-16t^2\rangle.\end{aligned}$$

(c) Obtain an equation for the parabola that is the path of the object.// An equivalent set of parametric equations for **r**(t) is x = 2t and $y = 1 + 2t - 16t^2$. Solving for t in the "x" equation and substituting the result into the equation for "y" provides us with $y = 1 + x - 4x^2$, an equation for the parabola.

5. (10 pts.) Obtain an arc-length parameterization for the curve **r**(t) = < cos(t), 2t, sin(t) > in terms of the initial point (1, 4π, 0). Rather than overloading the symbol **r**, write this new parameterization as **R**(s). How are **R** and **r** related? [Hint: What is the parameter t_0 that gives the initial point in vector form, i.e., **r**(t_0) = < 1, 4π, 0> ?? It's not zero, Folks.]

First, since **r**(2π) = <1, 4π, 0>, the (signed) distance from the point given by **r**(2π) to **r**(t) is

$$\begin{aligned} s = \varphi(t) &= \int_{2\pi}^{t} |\mathbf{r}'(u)|\, du \\ &= \int_{2\pi}^{t} |\langle -\sin(u), 2, \cos(u)\rangle|\, du \,. \\ &= \int_{2\pi}^{t} 5^{1/2}\, du = 5^{1/2}(t - 2\pi) \end{aligned}$$

Solving for t in terms of s yields

$$t = 5^{-1/2}s + 2\pi \ \ \textit{so that } \varphi^{-1}(t) = 5^{-1/2}t + 2\pi.$$

Thus,

$$\begin{aligned} \mathbf{R}(s) &= \mathbf{r}(\varphi^{-1}(s)) = \mathbf{r}(5^{-1/2}s+2\pi) \\ &= \langle\cos(5^{-1/2}s+2\pi), 2(5^{-1/2}s+2\pi), \sin(5^{-1/2}s+2\pi)\rangle. \end{aligned}$$

Evidently, **R**(s) = **r**(φ^{-1}(s)), or equivalently, **r**(t) = **R**(φ(t)). Also, you can simplify **R**(s) slightly. Why?

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6. (10 pts.) Let **r**(t) = <4·cos(t) , 4·sin(t), 3·t> for t $\varepsilon\mathbb{R}$. Then

(a) **r**′(t) = < -4·sin(t) , 4·cos(t), 3 > ,

(b) **r**″(t) = < -4·cos(t) , -4·sin(t), 0 > ,

(c) **T**(t) = < -(4/5)sin(t) , (4/5)cos(t), 3/5 > ,

(d) **N**(t) = < -cos(t) , -sin(t), 0 > , and

(e) κ(t) = | **T**′(t) |/| **r**′(t) | = 4/25 .

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7. (10 pts.) A particle moves smoothly in such a way that at a particular time t = 0, we have $\mathbf{r}(0) = < 0 , 1>$, $\mathbf{v}(0) = < 1 , 0>$ and $\mathbf{a}(0) = < 0 , -2>$. If we write $\mathbf{a}(0)$ in terms of $\mathbf{T}(0)$ and $\mathbf{N}(0)$, then

$$\mathbf{a}(0) = a_{\mathbf{T}}(0)\mathbf{T}(0) + a_{\mathbf{N}}(0)\mathbf{N}(0),$$

where

(a) $\mathbf{T}(0) = < 1, 0 >$,

(b) $a_{\mathbf{T}}(0) = \mathbf{T}(0)\cdot\mathbf{a}(0) = < 1, 0 >\cdot< 0,-2 > = 0$,

(c) $a_{\mathbf{N}}(0) = (|\mathbf{a}(0)|^2 - (a_{\mathbf{T}}(0))^2)^{1/2} = 2$, and

(d) $\mathbf{N}(0) = (1/2)[\mathbf{a}(0) - a_{\mathbf{T}}(0)\mathbf{T}(0)] = < 0, -1 >$.

(e) Consequently, an equation for the circle of curvature is given by:

$$x^2 + (y - (1/2))^2 = 1/4$$

since $\kappa(0) = |<1,0,0> \times <0,-2,0>|/|\mathbf{v}(0)|^3 = 2$, $\rho(0) = 1/2$, and $\mathbf{C}(0) = \mathbf{r}(0) + \rho(0)\mathbf{N}(0) = <0,1> + (1/2)<0,-1> = <0,1/2>$.

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8. (10 pts.) (a) Describe the graph of the equation

$$r = 4\cos(\theta) - 6\sin(\theta)$$

as precisely as possible. [We are playing in the cylindrical world, not merely on a polar ice floe.] You may wish to convert this equation to an equivalent rectangular equation to do this.

$$r = 4\cos(\theta) - 6\sin(\theta) \Rightarrow r^2 = 4r\cos(\theta) - 6r\sin(\theta)$$

$$\Rightarrow x^2 + y^2 = 4x - 6y$$

$$\Rightarrow (x - 2)^2 + (y + 3)^2 = 13.$$

Thus, we can see easily that the graph of the equation above is that of a cylinder with axis of symmetry given by the line in 3-space with parametric equations x = 2, y = -3, and z = t. Slicing the cylinder with any of the planes $z = z_0$ results in a circle with radius $13^{1/2}$.

(b) What are the precise spherical coordinates of the point in 3-space with rectangular coordinates (x, y, z) = (-1, -2, -3)? In doing this, ensure that $0 \le \theta < 2\pi$.
[Hint: You will have to express your answer using your friends, $\cos^{-1}$ and $\tan^{-1}$. The angles are rude today and not given to pious niceties.]

The spherical coordinates are given by

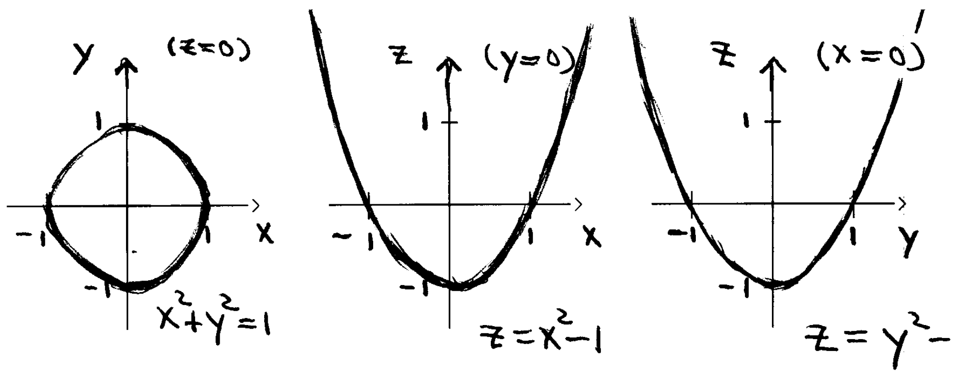
$$(\rho, \phi, \theta) = ((14)^{1/2}, \cos^{-1}(-3/(14)^{1/2}), \tan^{-1}(2) + \pi).$$

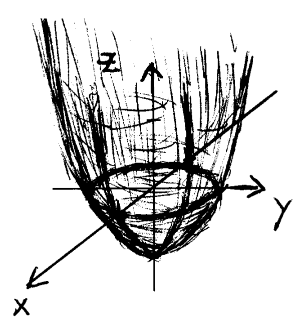
9. (10 pts.) Write an equation for the surface generated when the curve defined by the equation $x = 4 - y^2$ in the x,y - plane is revolved around the x - axis.

The curve in the xy-plane defined by $x = 4 - y^2$ may be realized as the zero set for the function $h(x,y) = 4 - y^2 - x$, that is the set of pairs (x,y) where $h(x,y) = 0$. [This is a formality to deal easily with the issue of uniform substitution!] The surface obtained when the curve is revolved around the x-axis consists of triples (x_0,y_0,z_0) with the property that the intersection of the surface with the plane $x = x_0$ parallel to the yz-plane is a point on a circle with center $(x_0,0,0)$ and radius r_0 obtained from the point (x_0,r_0) lying on the curve defined by $h(x,y) = 0$. Since (x_0,y_0,z_0) lies on the circle, $r_0 = ((y_0)^2 + (z_0)^2)^{1/2}$. Substituting (x_0,r_0) into $h(x,y) = 0$ yields $0 = 4 - ((y_0)^2 + (z_0)^2) - x_0$. Dropping the subscripts and doing a little algebra gives us the desired equation:

$$x = 4 - y^2 - z^2.$$

10. (10 pts.) Do the three 2-space sketches of the traces in each of the coordinate planes of the surface defined by $z = x^2 + y^2 - 1$. Work below and label carefully. Then on the back of page 4 attempt to do a 3 - space sketch in the plane of the surface.





Silly 10 Point Bonus: Your friendly garden variety parabola, $y = x^2$, is given in vector form by the equation

$$\boldsymbol{r}(t) = \langle t, t^2 \rangle,\ t \in \mathbb{R},$$

Perform the algebraic magic of obtaining the center of the osculating circle as a function of *t*. [Hint: You need the principal normal as a function of *t* and curvature, too.] Where’s your work? It won’t fit here!!