Silly 10 Point Bonus: Your friendly garden variety parabola, $y = x^2$, is given in vector form by the equation

$$\boldsymbol{r}(t) = \langle t, t^2 \rangle,\ t \in \mathbb{R},$$

Perform the algebraic magic of obtaining the center of the osculating circle as a function of *t*. [Hint: You need the principal normal as a function of *t* and curvature, too.] Where's your work? It won't fit here!! [On second thought, it might.]

First,

$$\boldsymbol{r}'(t) = \langle 1, 2t \rangle,\ \boldsymbol{r}''(t) = \langle 0, 2 \rangle, \textit{ and } |\boldsymbol{r}'(t)| = (1+4t^2)^{1/2} \textit{ for } t \in \mathbb{R}.$$

Thus,

$$\boldsymbol{T}(t) = (1+4t^2)^{-1/2}\langle 1, 2t \rangle.$$

Next,

$$\begin{aligned}\frac{d\boldsymbol{T}}{dt}(t) &= (-1/2)(1+4t^2)^{-3/2}(8t)\langle 1, 2t \rangle + (1+4t^2)^{-1/2}\langle 0, 2 \rangle \\ &= \frac{-4t}{(1+4t^2)^{3/2}}\langle 1, 2t \rangle + \frac{1+4t^2}{(1+4t^2)^{3/2}}\langle 0, 2 \rangle \\ &= \frac{1}{(1+4t^2)^{3/2}}[\langle -4t, -8t^2 \rangle + \langle 0, 2+8t^2 \rangle] \\ &= \frac{2}{(1+4t^2)^{3/2}}\langle -2t, 1 \rangle.\end{aligned}$$

Consequently,

$$\left|\frac{d\boldsymbol{T}}{dt}(t)\right| = \left|\frac{2}{(1+4t^2)^{3/2}}\langle -2t, 1 \rangle\right| = \frac{2}{1+4t^2},$$

$$\boldsymbol{N}(t) = \frac{1}{(1+4t^2)^{1/2}}\langle -2t, 1 \rangle, \textit{ and}$$

$$\kappa(t) = \frac{2}{(1+4t^2)^{3/2}}.$$

It is now easy to put the pieces together to locate the center via the usual vector equation:

$$\begin{aligned}\boldsymbol{C}(t) &= \boldsymbol{r}(t) + \rho(t)\boldsymbol{N}(t) \\ &= \langle t, t^2 \rangle + \frac{1+4t^2}{2}\langle -2t, 1 \rangle \\ &= \langle -4t^3, \frac{1}{2} + 3t^2 \rangle.\end{aligned}$$

Questions: How does knowledge that $\mathbf{T}\cdot\mathbf{T}' = 0$ help in transforming $\mathbf{T}'$?? This, of course, is where all the real work lies. It would appear that the point (0,1/2) is part of the formula for the center. Who is (0,1/2)?? Hmmm...