Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Vector objects must be denoted by using arrows. Do not "box" your final results. Show me all the magic on the page.

1. (10 pts.) (a) What is the largest possible domain of the function $f(x,y) = (9 - x^2 - y^2)^{1/2}$?

The domain of f is the set $D = \{ (x,y) : 9 - x^2 - y^2 \geq 0 \}$ which is the same as $\{ (x,y) : x^2 + y^2 \leq 9 \}$, the closed disk centered at (0,0) with radius 3.

(b) Consider the function $f(x,y) = x - y^2$. Obtain an equation for the level curve for this function that passes through the point (1,-2) in the x,y - plane. [Hint: What is the *level* for the level curve?]

The *level* is simply the function value. In this case, that turns out to be $f(1,-2) = -3$. Thus, the level curve through the point (1,-2) is the set of ordered pairs satisfying $f(x,y) = -3$, or more precisely, the set of ordered pairs satisfying $x - y^2 = -3$.

2. (10 pts.)
Let $f(x,y) = (x^2 + y^2)^{1/2}$. Compute the gradient of f at (-12,5), and then use it to compute $D_{\mathbf{u}}f(-12,5)$, where **u** is the unit vector in the same direction as $\mathbf{v} = \langle -4,3 \rangle$.

$$\nabla f(x,y) = \left\langle \frac{1}{2}(x^2 + y^2)^{-1/2}(2x), \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) \right\rangle$$

$$= \left\langle \frac{x}{(x^2 + y^2)^{1/2}}, \frac{y}{(x^2 + y^2)^{1/2}} \right\rangle$$

$$\nabla f(-12,5) = \left\langle -\frac{12}{13}, \frac{5}{13} \right\rangle$$

$$\boldsymbol{u} = \frac{1}{|\boldsymbol{v}|}\boldsymbol{v} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$$

$$D_{\mathbf{u}}f(-12,5) = \nabla f(-12,5) \cdot \boldsymbol{u} = \left\langle -\frac{12}{13}, \frac{5}{13} \right\rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = \frac{63}{65}.$$

3. (10 pts.) (a) If f(x,y) is a function of two variables, state the definition of the partial derivative of f with respect to x.// The partial derivative of f with respect to x is the function $f_x(x,y)$ defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

where the limit exists.

(b) Let $f(x,y) = x^2y^2 - 2y$. Using only the definition, reveal all the details of the computation of $f_x(x,y)$.

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2y^2-2y]-[x^2y^2-2y]}{h} \\ &= \lim_{h\to 0} \frac{[x^2y^2+2xhy^2+h^2y^2-2y]-[x^2y^2-2y]}{h} \\ &= \lim_{h\to} \frac{2xhy^2+h^2y^2}{h} \\ &= \lim_{h\to 0} (2xy^2 + hy^2) \\ &= 2xy^2. \end{aligned}$$

4. (10 pts.)

Let $f(x,y) = \tan^{-1}(2x+3y)$. (a) Compute the total differential, df, of the function f.

$$\begin{aligned} df &= f_x(x,y)\,dx + f_y(x,y)\,dy \\ &= \frac{2}{1+(2x+3y)^2}dx + \frac{3}{1+(2x+3y)^2}dy. \end{aligned}$$

(b) Use differentials to approximate the numerical value of f(.1,-.2) when f is the function of part (a) of this problem.

First observe that 2(.1)+3(-.2) = -.4 which is "near" 0, where we know $\tan^{-1}(0) = 0$. Consequently, there are a couple of natural choices for (x_0,y_0).

One possibility is $(x_0,y_0) = (0,0)$. Then let $(x_0 + \Delta x,\ y_0 + \Delta y) = (.1,-.2)$ so that $\Delta x = .1$ and $\Delta y = -.2$. Consequently,

$$f(.1,-.2) = f(x_0+\Delta x,\ y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

$$= 0 + (2)(.1) + (3)(-.2) = -.4.$$

A second reasonable choice is $(x_0,y_0) = (.3,-.2)$. Then $f(.3,-.2) = \tan^{-1}(2(.3) + 3(-.2)) = \tan^{-1}(0) = 0$. Again let $(x_0 + \Delta x,\ y_0 + \Delta y) = (.1,-.2)$ so that $\Delta x = -.2$ and $\Delta y = 0$. Consequently,

$$f(.1,-.2) = f(x_0+\Delta x,\ y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

$$= 0 + (2)(-.2) + (3)(0) = -.4.$$

Note: This is not really a good approximation!

5. (10 pts.) (a) Using complete sentences and appropriate notation, give the ε - δ definition for

(*) $$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

We say that the limit of f(x,y) is L as (x,y) approaches (a,b) and write (*) if L is a number such that, for each $\varepsilon > 0$, there is a number $\delta > 0$ with the following property: if (x,y) is any ordered pair in the domain of f with $0 < ((x-a)^2 + (y-b)^2)^{1/2} < \delta$, then it must follow that $|f(x,y) - L| < \varepsilon$.

(b) In the example where Edwards and Penney were showing that

$$\lim_{(x,y)\to(a,b)} xy = ab$$

they asserted that if they took $f(x,y) = x$ and $g(x,y) = y$, then it followed from the definition of limit that

(i) $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = a$ and (ii) $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = b$.

Choose one of equations (i) or (ii), indicate to me which you have chosen, and then prove the equation is true using the ε-δ definition.// We shall show (i). The proof of (ii) is similar. Thus, let $\varepsilon > 0$ be arbitrary. Now choose any $\delta > 0$ with $\delta \le \varepsilon$. We shall now verify that this δ does the dastardly deed. To this end, let (x,y) be any pair of real numbers with $0 < ((x-a)^2 + (y-b)^2)^{1/2} < \delta$. It then follows that

$$|f(x,y) - a| = |x - a| = ((x-a)^2)^{1/2}$$
$$\le ((x-a)^2 + (y-b)^2)^{1/2} < \delta \le \varepsilon.//$$

6. (10 pts.) Let $f(x,y) = \sin(3x + 4y)$ and let $P_0 = (0 , 0)$.
(a) Find a unit vector in the direction in which f(x,y) decreases most rapidly at P_0.

Since $\nabla f(x,y) = \langle 3\cos(3x+4y), 4\cos(3x+4y) \rangle$, the unit vector we want is the one opposite the direction of the vector $\nabla f(0,0) = \langle 3, 4 \rangle$. Thus we want the unit vector

$$\mathbf{u} = \left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle.$$

(b) Compute the rate of change of f(x,y) at P_0 in the direction in which f(x,y) increases most rapidly.

Since $\nabla f(0,0) = \langle 3, 4\rangle$, the rate we want is

$$|\nabla f(0,0)| = |\langle 3, 4 \rangle| = 5.$$

7. (10 pts.) Obtain an equation for the tangent plane to the surface defined by the equation $5x^2 + 6y^2 - 2z^2 = 9$ at the point (-1, 1,-1) which is on the surface.//

Solution 1: The surface is clearly the level surface through the point (-1, 1,-1) of the function $g(x,y,z) = 5x^2 + 6y^2 - 2z^2$. Consequently, the gradient of g at (-1, 1, -1) is perpendicular to the surface. Now $\nabla g(x,y,z) = \langle 10x,12y,-4z\rangle$. Thus, we have $\nabla g(-1,1,-1) = \langle -10,12,4\rangle$. An equation for the tangent plane we want is given by $-10(x - (-1)) + 12(y - 1) + 4(z -(-1)) = 0$.//

Solution 2: By pretending z is a function of the two independent variables x and y and performing partial differentiations on both sides of the equation above, we have

$$10x - 4z\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; \frac{\partial z}{\partial x} = \frac{5x}{2z}\text{, and}$$

$$12y - 4z\frac{\partial z}{\partial y} = 0 \;\Rightarrow\; \frac{\partial z}{\partial y} = \frac{3y}{z}\,.$$

Since $\left.\frac{\partial z}{\partial x}\right|_{(-1,1,-1)} = \frac{5}{2}$, and $\left.\frac{\partial z}{\partial y}\right|_{(-1,1,-1)} = -3$,

an equation for the plane is $z + 1 = (5/2)(x + 1) - 3(y - 1)$.//

NOTE: The implicit solution is legitimate since the hypotheses to Theorem 3 of Section 12.7 [13.7 Paperback] are satisfied. If you are feeling feisty, you may solve for z explicitly and really create a mess. That will get you a 3rd solution. Bon voyage!!

8. (10 pts.) Locate and classify the critical points of the function $f(x,y) = 2x^4 + y^2 - 4xy$. Use the second partials test in making your classification. (Fill in the table below after you locate all the critical points.)// Since $f_x(x,y) = 8x^3 - 4y$ and $f_y(x,y) = 2y - 4x$, the critical points of f are given by the solutions to the following system:

$$8x^3 - 4y = 0 \text{ and } 2y - 4x = 0.$$

The system is plainly equivalent to $2x^3 - y = 0$ and $y - 2x = 0$. Solving the system yields three critical points: (0,0), (1,2), and (-1,-2). [It’s easy. Just don’t lose any via division!!]

Crit.Pt.	f_{xx} @ c.p.	f_{yy} @ c.p.	f_{xy} @ c.p.	Δ @ c.p.	Conclusion
(x,y)	$24x^2$	2	-4	$48x^2 - 16$	
(0,0)	0	2	-4	-16	Saddle
(1,2)	24	2	-4	32	Rel.Min.
(-1,-2)	24	2	-4	32	Rel.Min.

9. (10 pts.) (a) Suppose that $y = h(x,z)$ satisfies the equation $F(x,y,z) = 0$, and that $F_y \neq 0$. Show how to compute $\partial y/\partial z$ in terms of the partial derivatives of F.// We'll write this twice, just for the fun of it. First, using classical curly d notation, since x and z are independent variables and y is a function of x and z, we have

$$0 = \frac{\partial F(x,y,z)}{\partial z} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial z} \Rightarrow 0 = \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial F}{\partial z}$$

$$\Rightarrow \frac{\partial y}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial y}.$$

Second, if we use subscript notation for the partial derivatives, we may set $g(x,z) = F(x, h(x,z), z)$. Then since $\partial x/\partial z = 0$,

$$0 = g_z(x,z) = F_y(x, h(x,z), z)h_z(x, z) + F_z(x, h(x,z), z),$$

implying $\partial y/\partial z = h_z(x,z) = -F_z(x, h(x,z), z)/F_y(x, h(x,z), z)$.

(b) Is f defined by $f(x,y) = \begin{cases} \dfrac{\sin(5(x^2+y^2))}{x^2+y^2} & , (x,y)\neq(0,0) \\ 5 & , (x,y)=(0,0) \end{cases}$

continuous at (0,0)? A complete explanation is required. Details & definitions are at the heart of it.// Since $f(0,0) = 5$ and

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(5(x^2+y^2))}{x^2+y^2}$$
$$= \lim_{r\to 0^+} \frac{\sin(5r^2)}{r^2}$$
$$\overset{(L'H)}{=} \lim_{r\to 0^+} \frac{10r\cos(5r^2)}{2r} = 5$$

by using L'Hopital's rule after doing a conversion to polar coordinates, f is continuous at (0,0).

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10. (10 pts.) Obtain and locate the absolute extrema of the function $f(x,y) = (x - y)^2$ on the closed disk D with radius five centered at the origin. Observe that D is given in set builder notation as follows: $D = \{ (x,y) : x^2 + y^2 \leq 25 \}$. In performing this magic, use Lagrange multipliers to deal with the behavior of f on the boundary, $B = \{ (x,y) : x^2 + y^2 = 25 \}$.
// Since $f_x(x,y) = 2(x - y)$ and $f_y(x,y) = -2(x - y)$, the critical points of f are the points of the form (x,x) in the interior of B. There we have $f(x,x) = 0$, the obvious minimum.

Set $g(x,y) = x^2 + y^2 - 25$. Then (x,y) is on the circle defined by $x^2 + y^2 = 25$ precisely when (x,y) satisfies $g(x,y) = 0$. Since $\nabla g(x,y) = \langle 2x, 2y\rangle$, $\nabla g(x,y) \neq \langle 0, 0\rangle$ when (x,y) is on the circle given by $g(x,y) = 0$. Plainly f and g are smooth enough to satisfy the hypotheses of the Lagrange Multiplier Theorem. Thus, if a constrained local extremum occurs at (x,y), there is a number λ so that $\nabla f(x,y) = \lambda\nabla g(x,y)$. Now $\nabla f(x,y) = \lambda\nabla g(x,y) \Rightarrow \langle 2(x-y),-2(x-y)\rangle = \lambda\langle 2x,2y\rangle$ $\Rightarrow x^2-y^2 = 0 \Rightarrow x = y$ or $x = -y$ by performing a little routine algebraic magic. Solving each of the systems consisting of (a) $x^2 + y^2 = 25$ and $x = y$, and (b) $x^2 + y^2 = 25$ and $x = -y$, yields $(5/2^{1/2},5/2^{1/2})$, $(-5/2^{1/2},-5/2^{1/2})$, $(5/2^{1/2},-5/2^{1/2})$, and $(-5/2^{1/2},5/2^{1/2})$. It turns out f is zero, the minimum value, at the first two points, and f is 50, the maximum value, at the last two points.

Silly 10 Point Bonus: Suppose that $(x_0,y_0) \neq (0,0)$. Without using any of the usual tools of Calculus, like the derivative, obtain the absolute extrema of the function

$$f(\theta) = x_0\cos(\theta) + y_0\sin(\theta)$$

and locate the precise θ's in the interval $[0,2\pi)$ where the extrema occur.
[Indicate where your work is.]