Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Vector objects must be denoted by using arrows. Do not "box" your final results. Show me all the magic on the page.

Silly 10 Point Bonus: Suppose that $(x_0,y_0) \neq (0,0)$. Without using any of the usual tools of Calculus, like the derivative, obtain the absolute extrema of the function

$$f(\theta) = x_0\cos(\theta) + y_0\sin(\theta)$$

and locate the precise θ's in the interval $[0,2\pi)$ where the extrema occur. [Indicate where your work is.]

Parts of this are easy if you realize that

$$\begin{aligned} f(\theta) &= x_0\cos(\theta) + y_0\sin(\theta) \\ &= \langle x_0,y_0\rangle\cdot\langle\cos(\theta),\sin(\theta)\rangle \\ &= |\langle x_0,y_0\rangle|\cos(\varphi) \end{aligned}$$

where φ is the angle between the nonzero vector $\langle x_0, y_0\rangle$ and the unit vector **u** = $\langle \cos(\theta), \sin(\theta)\rangle$. In fact, the extreme values may be read off of the equation above. The maximum value that f attains is

$$|\langle x_0,y_0\rangle| = ((x_0)^2+(y_0)^2)^{1/2}$$

and the minimum value is

$$-|\langle x_0,y_0\rangle| = -((x_0)^2+(y_0)^2)^{1/2}.$$

The real problem here is in locating where the extrema occur in terms of θ in the interval $[0,2\pi)$.

Plainly, the maximum occurs when $\varphi = 0$ and **u** has the same direction as $\langle x_0, y_0\rangle$. Thus, we need only find the solution in the interval $[0,2\pi)$ to the system

$$\cos(\theta) = \frac{x_0}{((x_0)^2+(y_0)^2)^{1/2}}$$

and

$$\sin(\theta) = \frac{y_0}{((x_0)^2+(y_0)^2)^{1/2}}.$$

This is easy once you think about converting the point (x_0, y_0) to polar coordinates. A solution:

$$\theta_0 = \begin{cases} \pi/2 & , \text{ if } x_0 = 0 \text{ and } y_0 > 0 ; \\ 3\pi/2 & , \text{ if } x_0 = 0 \text{ and } y_0 < 0 ; \\ \arctan(y_0/x_0) & , \text{ if } x_0 > 0 \text{ and } y_0 \geq 0 ; \\ \arctan(y_0/x_0) + \pi & , \text{ if } x_0 < 0 \text{ and } y_0 \in \mathbb{R} ; \\ \arctan(y_0/x_0) + 2\pi & , \text{ if } x_0 > 0 \text{ and } y_0 < 0 . \end{cases}$$

The minimum, of course, occurs when $\varphi = \pi$ and **u** has the opposite direction from $\langle x_0, y_0\rangle$. Let's denote the value of

θ in the interval $[0,2\pi)$ yielding the minimum by θ_1. Then θ_1 may be given in terms of the values for θ_0 as follows:

$$\theta_1 = \begin{cases} \theta_0 + \pi\ , & \text{if } y_0 > 0\ ; \\ \theta_0 - \pi\ , & \text{if } y_0 < 0\ ; \\ \theta_0 + \pi\ , & \text{if } y_0 = 0 \text{ and } x_0 > 0; \\ \theta_0 - \pi\ , & \text{if } y_0 = 0 \text{ and } x_0 < 0\ . \end{cases}$$

Observe that the stickiness is caused by the demand that we keep both in the interval $[0.2\pi)$ so that one cannot simply add π or subtract π without regard for where (x_0, y_0) is. What is going on is this, of course: The point (x_0, y_0) provides the angle for the maximum value. Thus, if (x_0, y_0) is above the x-axis, add π to θ_0 to get the angle for the minimum. If (x_0, y_0) is below the x-axis, subtract π from θ_0 to get the angle for the minimum. If (x_0, y_0) lies on the positive x-axis, add π to θ_0 to get the angle for the minimum. And finally, if (x_0, y_0) is on the negative x-axis, subtract π from θ_0 to get the angle for the minimum. Rats!

Here is one last thing for you to puzzle over. Observe that φ is the angle between the fixed, nonzero vector $< x_0, y_0 >$ and **u**, and θ may be viewed as the measure of the angle formed by the unit vector $< \cos(\theta),\sin(\theta) >$ and the unit vector **i**. Except when $< x_0, y_0 >$ is a positive multiple of $< 1\ , 0 >$, φ and θ are different. Are φ and θ above related? How?? Answering this points to an alternative solution of the original problem, of course.

Note: This nonsense is inspired by the usual analysis relating the gradient vector to the directional derivative. Were you paying attention??

$$\begin{aligned} D_{\mathbf{u}}f(x_0,y_0) &= \nabla f(x_0,y_0)\cdot \boldsymbol{u} \\ &= \langle\ f_x(x_0,y_0),\ f_y(x_0,y_0)\ \rangle \cdot \langle\ u_1,\ u_2\ \rangle \\ &= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2 \\ &= |\nabla f(x_0,y_0)|\cos(\varphi), \end{aligned}$$

where φ is the angle between the nonzero gradient and the unit vector **u**. Look at the pattern.