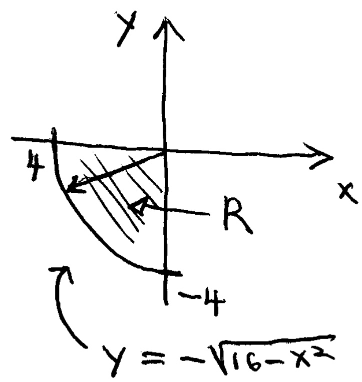
Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Generic vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page neatly.

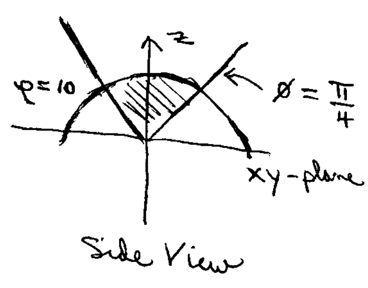
1. (10 pts.) Convert the given iterated integral into an iterated integral in polar coordinates that has the same numerical value and is easier to evaluate, perhaps. Do not attempt to evaluate the polar integral. A picture might help.

$$\int_{-4}^{0}\int_{-(16-x^2)^{1/2}}^{0} e^{(x^2+y^2)}\ dy\,dx = \int_{\pi}^{3\pi/2}\int_{0}^{4} re^{r^2}\,dr\,d\theta.$$



2. (10 pts.) Write down, but do not attempt to evaluate the triple iterated integral in spherical coordinates that provides the volume of the solid T that is bounded above by the sphere defined by $x^2 + y^2 + z^2 = 100$ and below by the cone defined by $z = (x^2 + y^2)^{1/2}$.

$$\iiint_T 1\ dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{10} \rho^2\sin\phi\ d\rho\,d\phi\,d\theta.$$



3. (10 pts.)
Let **F**(x,y,z) = <xy^2, yz^2, zx^2>. Compute the divergence and the curl of the vector field **F**.

(a) $$\mathit{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(zx^2) = y^2 + z^2 + x^2.$$

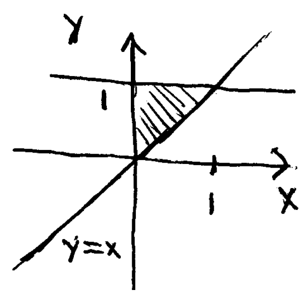
(b)

$$\mathit{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & yz^2 & zx^2 \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(zx^2) - \frac{\partial}{\partial z}(yz^2), -\left(\frac{\partial}{\partial x}(zx^2) - \frac{\partial}{\partial z}(xy^2)\right), \frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial y}(xy^2) \right\rangle$$

$$= \langle -2yz, -2xz, -2xy \rangle.$$

4. (10 pts.) Compute the surface area of the part of the graph of the surface defined by $z = x + y^2$ that lies above the triangle in the xy-plane given by the pairs (x,y) that satisfy $0 \le x \le 1$ and $x \le y \le 1$. [Of course *above* means $z \ge 0$. One order of integration may be easier than the other.]



$$SA = \iint_R \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 \right)^{1/2} dA$$

$$= \iint_R ((1)^2 + (2y)^2 + 1)^{1/2}\, dA$$

$$= \iint_R (2 + 4y^2))^{1/2}\, dA$$

$$= \int_0^1 \int_x^1 (2+4y^2)^{1/2}\, dy\,dx \quad \textit{HARD!!}$$

$$= \int_0^1 \int_0^y (2+4y^2)^{1/2}\, dx\,dy \quad \textit{Easier...}$$

$$= \int_0^1 y(2+4y^2)^{1/2}\, dy$$

$$= \int_2^6 u^{1/2} \cdot \frac{1}{8}\, du$$

$$= \frac{1}{8}\left(\frac{2}{3}u^{3/2}\right)\Big|_2^6 = \frac{1}{12}((6)^{3/2} - (2)^{3/2}).$$

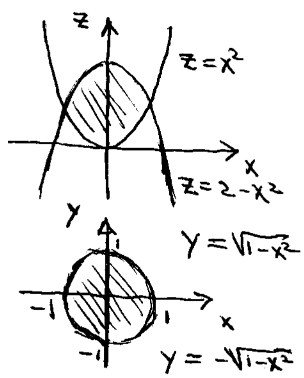
Obviously, we used the u-substitution $u = 2 + 4y^2$ above. The *hard* route leads tangentially to an ugly $\sec^3$ integration if done correctly.

5. (10 pts.) Write down a triple iterated integral in cartesian coordinates that would be used to evaluate

$$\iiint_T f(x,y,z)\ dV\ ,$$

where f(x,y,z) = x + y and T is the region contained between the two paraboloids $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$, but do not attempt to evaluate the triple iterated integral you have obtained. [Sketching the traces in the coordinate planes might help. Determining the projection of the intersection of the two surfaces on the xy-plane is essential.]

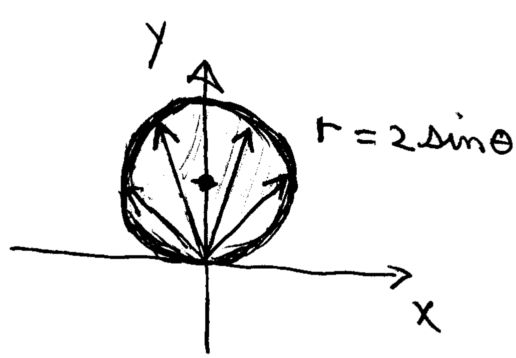
$$\iiint_T f(x,y,z)\ dV = \int_{-1}^{1}\int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}}\int_{x^2+y^2}^{2-x^2-y^2} x + y\ \ dz\,dy\,dx$$
$$= \int_{-1}^{1}\int_{-(1-y^2)^{1/2}}^{(1-y^2)^{1/2}}\int_{x^2+y^2}^{2-x^2-y^2} x + y\ \ dz\,dx\,dy.$$



There are two obvious and easy natural orders of integration for this varmint since the projection on the xy-plane of the two surfaces is the unit circle centered at the origin. Of course you might try producing the other iterated integral.

6. (10 pts.) Write down the triple iterated integral in cylindrical coordinates that provides the numerical value of the volume of the region in 3-space bounded above and below by the spherical surface $r^2 + z^2 = 36$ and laterally by the cylinder $r = 2\cdot\sin(\theta)$, but do not attempt to evaluate the integral you obtain.

$$\iiint_T 1\ dV = \int_0^{\pi}\int_0^{2\sin(\theta)}\int_{-(36-r^2)^{1/2}}^{(36-r^2)^{1/2}} r\ \ dz\,dr\,d\theta.$$



$$r = 2\cdot\sin(\theta) \iff r^2 = 2r\cdot\sin(\theta)$$
$$\iff x^2 + y^2 = 2y$$
$$\iff x^2 + (y-1)^2 = 1.$$

By sketching the rectangular auxiliary rθ-graph, you can see that the circle is traced out as θ runs from 0 to π.

7. (10 pts.) Let

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

for (x,y) ≠ (0,0). (a) Fix ε > 0. If C is the path traced out by the vector-valued function $\boldsymbol{r}(t) = <\varepsilon\cos(t), \varepsilon\sin(t)>$, $t \in [0,2\pi]$, in the positive direction, evaluate the path integral,

$$\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} \frac{(-\varepsilon\sin(t))(-\varepsilon\sin(t))+(\varepsilon\cos(t))(\varepsilon\cos(t))}{(\varepsilon\cos(t))^2+(\varepsilon\sin(t))^2}\, dt$$

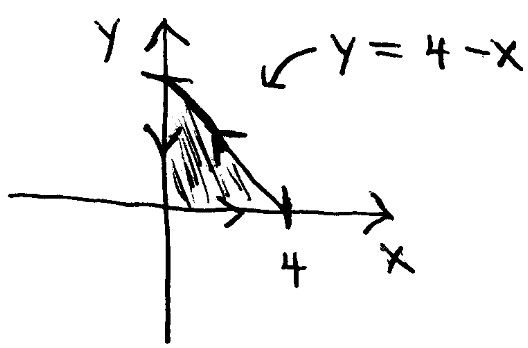
$$= \int_0^{2\pi} 1\, dt = 2\pi,$$

after the usual pythagorean noise.

(b) Explain briefly why your computation in part (a) demonstrates that **F** is not a conservative field on the plane with the origin, (0,0), removed.

Were **F** a conservative field on the plane with the origin, (0,0), removed, we would have had ∇φ(x,y) = **F**(x,y) for some function φ(x,y) defined for all (x,y) in the plane with the origin removed. It would then follow from the Fundamental Theorem of Path integrals that the path integral would have been zero since the path begins and ends at the same point in the plane. Clearly, this is not so, though.

8. (10 pts.) Starting at the point (0,0), a particle goes along the x-axis until it reaches the point (4,0). It then goes from (4,0) to (0,4) along a straight line. Finally the particle returns to the origin by travelling along the y-axis. Use Green's Theorem to compute the work done on the particle by the force field defined by **F**(x,y) = < -5y, 5x > for (x,y) ε $\mathbb{R}^2$. [Draw a picture. This is easy??]



$$W = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_C -5y\, dx + 5x\, dy$$

$$= \iint_R \frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(-5y)\, dA$$

$$= 10\iint_R 1\, dA = 10\cdot Area(\mathrm{R}) = 80,$$

Of course, if you forgot the formula for the area of a right triangle, you could turn the double integral into an iterated integral or two, thus:

$$\iint_R 1\, dA = \int_0^4\int_0^{4-x} 1\, dy\,dx = \int_0^4 4-x\, dx = (4x - \frac{x^2}{2})\Big|_0^4 = 8.$$

9. (10 pts.) (a) Show that the vector field

$$\mathbf{F}(x,y) = < 2xe^y,\ x^2e^y + 2y >$$

is actually a gradient field by producing a function $\phi(x,y)$ such that $\nabla\phi(x,y) = \mathbf{F}(x,y)$ for all (x,y) in the plane.

Evidently $\dfrac{\partial}{\partial y}(2xe^y) = 2xe^y = \dfrac{\partial}{\partial x}(x^2e^y + 2y)$ *for each* $(x,y) \in \mathbb{R}^2$.

Then

$$\frac{\partial\phi}{\partial x}(x,y) = 2xe^y \Rightarrow \phi(x,y) = \int 2xe^y\,dx = x^2e^y + c(y).$$

Therefore,

$$x^2e^y + 2y = \frac{\partial\phi}{\partial y}(x,y) = \frac{\partial}{\partial y}(x^2e^y) + \frac{dc}{dy}(y) \Rightarrow \frac{dc}{dy}(y) = 2y$$

$$\Rightarrow c(y) = y^2 + c_0$$

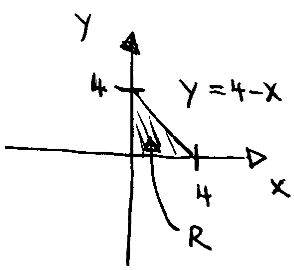
for some number c_0. Hence, $\phi(x,y) = x^2e^y + y^2 + c_0$.

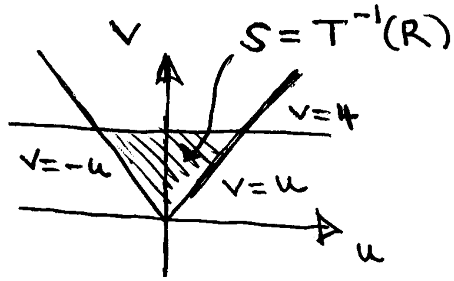
(b) Using the Fundamental Theorem of Line Integrals, evaluate the path integral below, where C is any smooth path from the origin to the point (1,1). [WARNING: You must use the theorem to get any credit here.]

$$\int_C 2xe^y dx + (x^2e^y + 2y)dy = (x^2e^x + y^2)\Big|_{(0,0)}^{(1,1)} = e + 1.$$

10. (10 pts.) Use the substitution $u = y - x$ and $v = y + x$ to evaluate the integral below, where R is the bound region enclosed by the triangle with vertices at (0,0), (4,0), and (0,4).

$$\begin{aligned}\iint_R y+x\,dx\,dy &= \iint_R y+x\,dA_{x,y}\\ &= \iint_S v\,\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dA_{u,v}\\ &= \int_0^4\int_{-v}^{v} v\,\left|-\frac{1}{2}\right|\,du\,dv\\ &= \int_0^4 \frac{v}{2}\int_{-v}^{v} 1\,du\,dv\\ &= \int_0^4 v^2\,dv = \left(\frac{v^3}{3}\right)\Big|_0^4 = \frac{64}{3}.\end{aligned}$$





T^{-1}: $u = y-x$, $v = y+x$

T: $x = (v-u)/2$,
$y = (u+v)/2$

Bounding Curves:

$x = 0 \Leftrightarrow u = v$
$y = 0 \Leftrightarrow u = -v$
$y = 4 - x \Leftrightarrow v = 4$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{vmatrix} = (-1/2)(1/2)-(1/2)(1/2) = -1/2.$$

Silly 10 Point Bonus: In Problem 7 you proved that the vector field

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$$

is not conservative on the plane with the origin removed. Despite this, obtain two potential functions for **F** on the lower half-plane consisting of LH = { $(x,y) \in \mathbb{R}^2$: $y < 0$ } and the right half-plane RH = { $(x,y) \in \mathbb{R}^2$: $0 < x$ }. Which potential goes with which half-plane??? Say where your work is, for it won't fit here. [It may be found in c3-t3-b.pdf.]