Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". *Vector objects must be denoted by using arrows.* Since a correct answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page clearly.

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1. (10 pts.) Suppose that an object is moving in a fixed plane with its acceleration given by **a**(t) = <0,-32>. Suppose that the initial position of the object is **r**(0) = <0,1> and the initial velocity of the object is **v**(0) = < 2, 2>.

(a) (6 pts.) Find the velocity of the object, **v**(t), as a function of time.// It follows easily from the vector-valued version of the Fundamental Theorem of Calculus that

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(u)\,du + \langle 2,2\rangle$$
$$= \int_0^t \langle 0,-32\rangle\,du + \langle 2,2\rangle = \langle 0,-32t\rangle + \langle 2,2\rangle = \langle 2,2-32t\rangle.$$

(b) (2 pts.) Find the position of the object, **r**(t), as a function of time.// Similarly,

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(u)\,du + \langle 0,1\rangle$$
$$= \int_0^t \langle 2,2-32u\rangle\,du + \langle 0,1\rangle = \langle 2t,2t-16t^2\rangle + \langle 0,1\rangle$$
$$= \langle 2t,\ 1+2t-16t^2\rangle.$$

(c) (2 pts.) Obtain an equation for the parabola that is the path of the object.
// An equivalent set of parametric equations for **r**(t) is $x = 2t$ and $y = 1 + 2t - 16t^2$. Solving for t in the "x" equation and substituting the result into the equation for "y" provides us with $y = 1 + x - 4x^2$, an equation for the parabola.

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2. (10 pts.) Obtain an equation for the plane that contains the points (1,0,0), (1,-2,0), and (1,-2,3).

Our main job is to build a normal vector for the plane. To this end, set $A = (1,0,0)$, $B = (1,-2,0)$, and $C = (1,-2,3)$. Then build a couple of non-parallel vectors in the plane by letting

$$\vec{AB} = \langle 0,-2,0\rangle \text{ and } \vec{AC} = \langle 0,-2,3\rangle.$$

A nonzero vector perpendicular to these two is given by

$$\vec{n} = \vec{AB}\times\vec{AC} = \langle 0,-2,0\rangle \times \langle 0,-2,3\rangle = \langle -6,0,0\rangle.$$

Consequently, an equation for the plane is simply $-6(x - 1) = 0$, or $x = 1$.

3. (10 pts.) Let **r**(t) = < 3cos(t) , 3sin(t), 4t > for $t \in \mathbb{R}$. Then

(a) **r**′(t) = < (-3)sin(t) , (3)·cos(t), 4 > ,

(b) **r**″(t) = < (-3)cos(t) , (-3)sin(t), 0 > ,

(c) **T**(t) = < -(3/5)sin(t) , (3/5)cos(t), 4/5 > ,

(d) **N**(t) = < -cos(t) , -sin(t), 0 > , and

(e) κ(t) = | **T**′(t) |/| **r**′(t) | = 3/25 .

4. (10 pts.) (a) Determine the radius and the center of the sphere defined by the following equation:

$$x^2 + y^2 + z^2 + 8x - 6y + 10z = 0$$

Completing the square thrice yields

$$(x + 4)^2 + (y - 3)^2 + (z + 5)^2 = 50.$$

Thus, the center is (-4,3,-5) and the radius is $50^{1/2} = 5(2^{1/2})$.

(b) What is the parameter t_0 of the point on the line defined by the vector equation < x, y, z > = t< 1, 1, 1> that is nearest to the point (1,3,2)?

The distance from the point (1,3,2) to the point on the line with parameter t is given by the length of the vector function

v(t) = < t - 1, t - 3, t - 2 >

built using <1,3,2> as the initial point and taking as the terminal point the point on the line with parameter value t. The distance is shortest for t_0 where **v**(t_0) is perpendicular to the direction vector for the line <1,1,1>. Thus we want the t_0 satisfying **v**(t_0)·<1,1,1> = 0. Computing this dot product and solving for t_0 in the little linear equation that results yields $t_0 = 2$.

This part of Problem 4 may also be done with elementary Calculus I techniques.

5. (10 pts.) Obtain an arc-length parameterization for the curve **r**(t) = < 3t, cos(t), sin(t) > in terms of the initial point (0, 1, 0). Rather than overloading the symbol **r**, write this new parameterization as **R**(s). How are **R** and **r** related?

First, since **r**(0) = <0, 1, 0>, the (signed) distance along the curve from the point given by **r**(0) on the graph to that given by **r**(t) is

$$\begin{aligned} s = \varphi(t) &= \int_0^t |\mathbf{r}'(u)|\, du \\ &= \int_0^t |\langle 3, -\sin(u), \cos(u)\rangle|\, du \\ &= \int_0^t 10^{1/2}\, du = 10^{1/2}(t - 0) = 10^{1/2} t. \end{aligned}$$

Solving for t in terms of s yields

$$t = 10^{-1/2} s \quad \textit{so that } \varphi^{-1}(t) = 10^{-1/2} t.$$

Thus,

$$\begin{aligned} \mathbf{R}(s) &= \mathbf{r}(\varphi^{-1}(s)) = \mathbf{r}(10^{-1/2} s) \\ &= \left\langle \frac{3}{10^{1/2}} s \,,\ \cos\left(\frac{s}{10^{1/2}}\right) ,\ \sin\left(\frac{s}{10^{1/2}}\right) \right\rangle. \end{aligned}$$

Evidently, **R**(s) = **r**(φ^{-1}(s)), or equivalently, **r**(t) = **R**(φ(t)).

6. (10 pts.) Obtain parametric equations for the line of intersection of the two planes defined by x + 3y - 6z = 10 and y + 2z = 5.

$$\begin{cases} x + 3y - 6z = 10 \\ \quad\ \ y + 2z = 5 \end{cases}$$

is equivalent to

$$\begin{cases} x \quad = -5 + 12z \\ \quad y = 5 - 2z. \end{cases}$$

This was obtained by doing standard "row" or "equation" operations. A set of parametric equations:

$$\begin{cases} x = -5 + 12t \\ y = 5 - 2t \\ z = t \ ,\ t \in \mathbb{R}. \end{cases}$$

Kindly note that there are several legitimate ways to reach a correct system of parametric equations, and there are infinitely many such systems. You can check by substitution into the equations for the planes.

7. (10 pts.) A particle moves smoothly in such a way that at a particular time t = 0, we have **r**(0) = < 1 , 0>, **v**(0) = < 1 , 1> and **a**(0) = < 2 , -1>. If we write **a**(0) in terms of **T**(0) and **N**(0), then

$$\mathbf{a}(0) = a_{\mathbf{T}}(0)\mathbf{T}(0) + a_{\mathbf{N}}(0)\mathbf{N}(0),$$

where

(a) **T**(0) = $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$,

(b) $a_{\mathbf{T}}(0)$ = $\frac{1}{\sqrt{2}}$,

(c) $a_{\mathbf{N}}(0)$ = $\frac{3}{\sqrt{2}}$, and

(d) **N**(0) = $\left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$.

(e) Consequently, an equation for the circle of curvature is given by:

$$\left(x - \tfrac{5}{3}\right)^2 + \left(y + \tfrac{2}{3}\right)^2 = \tfrac{8}{9}$$

To obtain this, you need both

$$\kappa(0) = \frac{3}{2\sqrt{2}}, \quad \text{and} \quad \mathbf{C}(0) = \mathbf{r}(0) + \rho(0)\mathbf{N}(0) = \left\langle \tfrac{5}{3}, \tfrac{-2}{3} \right\rangle.$$

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8. (10 pts.) (a) Describe the graph of the equation

$$r = 10\cos(\theta) + 6\sin(\theta)$$

as precisely as possible. [We are playing in the cylindrical world, not merely on a polar ice floe.] You may wish to convert this equation to an equivalent rectangular equation to do this.

$$r = 10\cos(\theta) + 6\sin(\theta) \Rightarrow r^2 = 10r\cos(\theta) + 6r\sin(\theta)$$
$$\Rightarrow x^2 + y^2 = 10x + 6y$$
$$\Rightarrow (x - 5)^2 + (y - 3)^2 = 34.$$

Thus, we can see easily that the graph of the equation above is that of a cylinder with axis of symmetry given by the line in 3-space with parametric equations x = 5, y = 3, and z = t. Slicing the cylinder with any of the planes $z = z_0$ results in a circle with radius $34^{1/2}$.

(b) What are the precise spherical coordinates of the point in 3-space with rectangular coordinates (x, y, z) = (-1, -2, -1)? In doing this, ensure that $0 \le \theta < 2\pi$. [Hint: You will have to express your answer using your friends, $\cos^{-1}$ and $\tan^{-1}$. The angles are rude today and not given to pi-ous niceties.]

$$(\rho, \phi, \theta) = \left((6)^{1/2}, \cos^{-1}(-1/(6)^{1/2}), \tan^{-1}(2) + \pi \right).$$

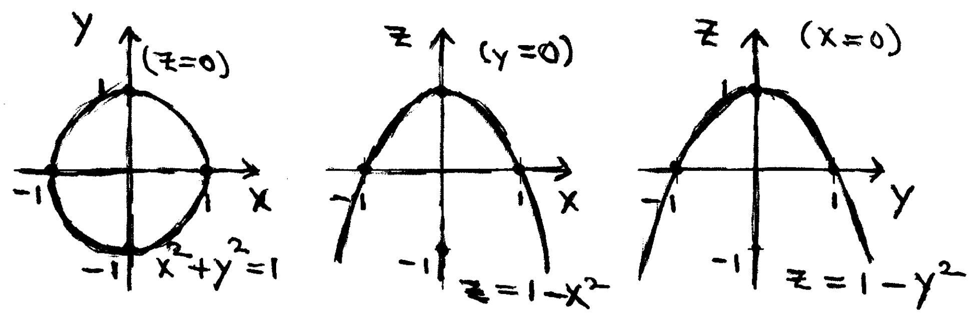
9. (10 pts.) (a) Find the direction angles of the vector ***v*** = < -1, 3, -2 >.

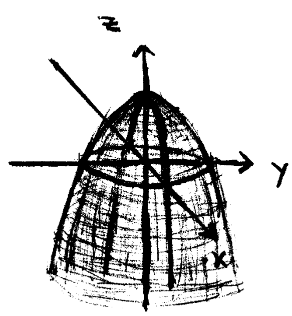
$$\alpha = \cos^{-1}\left(\frac{-1}{\sqrt{14}}\right), \quad \beta = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right), \quad \textit{and} \quad \gamma = \cos^{-1}\left(\frac{-2}{\sqrt{14}}\right).$$

(b) What is the angle θ between the two vectors ***v***, from part (a) above, and ***w*** = <1,2,1> ?

$$\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{84}}\right) = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right).$$

10. (10 pts.) Do the three 2-space sketches of the traces in each of the coordinate planes of the surface defined by $z = 1 - x^2 - y^2$. Work below and label very carefully. Then on the back of Page 4 attempt to do a 3 - space sketch in the plane of the surface.





Silly 10 Point Bonus: Plainly,

$$\begin{cases} x = 3 + \cos(t) \\ y = -4 + \sin(t)\ , \quad t \in [0, 2\pi] \end{cases}$$

and

$$\begin{cases} \arccos(x-3) = t \\ y = -4 + \sin(\arccos(x-3)) \end{cases}$$

are not equivalent. How must the second parameterization be altered so as to yield the same curve as that defined by the first parameterization?? [Say where your work is, for it won't fit here.]