**Read Me First:** Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. .... Vector objects must be denoted by using arrows. Do not "box" your final results. Show me all the magic on the page.

1. (10 pts.) (a) If f(x,y) is a function of two variables, using a complete sentence and appropriate notation, state the definition of the partial derivative of f with respect to y.
// The partial derivative of f with respect to y is the function $f_y(x,y)$ defined by

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

where the limit exists. //

(b) Let $f(x,y) = x^2y^2 - 2y$. Using only the definition, reveal all the details of the computation of $f_y(x,y)$.

$$\begin{aligned} f_y(x,y) &= \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k} \\ &= \lim_{k\to 0} \frac{[x^2(y+k)^2-2(y+k)]-[x^2y^2-2y]}{k} \\ &= \lim_{k\to 0} \frac{[x^2y^2+2x^2ky+k^2x^2-2y-2k]-[x^2y^2-2y]}{k} \\ &= \lim_{k\to 0} \frac{2x^2ky+k^2x^2-2k}{k} \\ &= \lim_{k\to 0} (2x^2y + kx^2 - 2) \\ &= 2x^2y - 2. \end{aligned}$$

2. (10 pts.) Let $f(x,y) = \sin(2x+3y)$. (a) Compute the total differential, df, of the function f.

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy = 2\cos(2x+3y)\,dx + 3\cos(2x+3y)\,dy.$$

(b) Use a linear approximation to approximate the numerical value of f(.1,-.1) when f is the function of part (a) of this problem.
// First observe that 2(.1)+3(-.1) = -.1 which is "near" 0, where we know sin(0) = 0. Consequently, there are a couple of natural choices for $(x_0,y_0)$.

One possibility is $(x_0,y_0) = (0,0)$. Then let $(x_0 + \Delta x,\ y_0 + \Delta y) = (.1,-.1)$ so that $\Delta x = .1$ and $\Delta y = -.1$. Consequently,

$$f(.1,-.1) = f(x_0+\Delta x,\ y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

$$= 0 + (2)(.1) + (3)(-.1) = -.1.$$

A second reasonable choice is $(x_0,y_0) = (.3,-.2)$. Then f(.3,-.2) = sin(2(.3) + 3(-.2)) = sin(0) = 0. Again let $(x_0 + \Delta x,\ y_0 + \Delta y) = (.1,-.1)$ so that $\Delta x = -.2$ and $\Delta y = .1$. Consequently, as before,

$$f(.1,-.1) \approx 0 + (2)(-.2) + (3)(.1) = -.1$$

3. (10 pts.) (a) Suppose that $x = h(y,z)$ satisfies the equation $F(x,y,z) = 0$, and that $F_x \neq 0$. Show how to compute $\partial x/\partial y$ in terms of the partial derivatives of F.
// We'll write this twice, just for the fun of it. First, using classical curly d notation, since $y$ and $z$ are independent variables and $x$ is a function of $y$ and $z$, we have

$$0 = \frac{\partial F(x,y,z)}{\partial y} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} \Rightarrow 0 = \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}$$

$$\Rightarrow \frac{\partial x}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial x}.$$

Second, if we use subscript notation for the partial derivatives, we may set $g(y,z) = F(h(y,z),y,z)$. Then since $\partial z/\partial y = 0$,

$$0 = g_y(y,z) = F_x(h(y,z),y,z)h_y(y,z) + F_y(h(y,z),y,z),$$

implying $\partial x/\partial y = h_y(y,z) = -F_y(h(y,z),y,z)/F_x(h(y,z),y,z)$.

(b) Is f defined by $f(x,y) = \begin{cases} \dfrac{\tan(\pi(x^2+y^2))}{2(x^2+y^2)} & , (x,y)\neq(0,0) \\ \pi & , (x,y)=(0,0) \end{cases}$

continuous at (0,0)? A complete explanation is required. Details & definitions are at the heart of it.
// Since $f(0,0) = \pi$ and

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{\tan(\pi(x^2+y^2))}{2(x^2+y^2)} \\ &= \lim_{r\to 0^+} \frac{\tan(\pi r^2)}{2r^2} \\ &\overset{(L'H)}{=} \lim_{r\to 0^+} \frac{2\pi r\sec^2(\pi r^2)}{4r} = \frac{\pi}{2} \end{aligned}$$

by using L'Hopital's rule after doing a conversion to polar coordinates, f is not continuous at (0,0).

4. (10 pts.) (a)Consider the function f of three variables defined by $f(x,y,z) = 5x^2 + 6y^2 - 4z^2$. Obtain an equation for the level surface for this function that passes through the point (1,-1, -1) in the 3 - space.

The *level* is simply the function value. In this case, that turns out to be $f(1,-1,-1) = 7$. Thus, the level surface through the point (1,-1,-1) is the set of ordered triples satisfying $f(x,y,z) = 7$ , or more precisely, the set of ordered triples satisfying $5x^2 + 6y^2 - 4z^2 = 7$.

(b) Obtain an equation for the plane that is tangent to the level surface given in part (a) with the point of tangency being (1,-1, -1).

The gradient of f at (1,-1,-1) is perpendicular to the surface and the tangent plane. Now $\nabla f(x,y,z) = \langle 10x,12y,-8z\rangle$. Thus, we have $\nabla g(-1,1,-1) = \langle 10,-12,8\rangle$. An equation for the tangent plane we want is given by $10(x - 1) - 12(y + 1) + 8(z + 1) = 0$.// This problem may also be done by two other means!!

5. (10 pts.) (a) Using complete sentences and appropriate notation, give the $\varepsilon$ - $\delta$ definition for

(*) $$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

We say that the limit of f(x,y) is L as (x,y) approaches (a,b) and write (*) if L is a number such that, for each $\varepsilon > 0$, there is a number $\delta > 0$ with the following property: if (x,y) is in the domain of f with $0 < ((x-a)^2 + (y-b)^2)^{1/2} < \delta$, then it must follow that $|f(x,y) - L| < \varepsilon$.

(b) In the example where Edwards and Penney were showing that

$$\lim_{(x,y)\to(a,b)} xy = ab$$

they asserted that if they took $f(x,y) = x$ and $g(x,y) = y$, then it followed from the definition of limit that

(i) $\lim_{(x,y)\to(a,b)} f(x,y) = a$ and (ii) $\lim_{(x,y)\to(a,b)} g(x,y) = b$.

Choose one of equations (i) or (ii), indicate to me which you have chosen, and then prove the equation is true using the $\varepsilon$-$\delta$ definition.

We shall show (ii). The proof of (i) is similar. Thus, let $\varepsilon > 0$ be arbitrary. Now choose any $\delta > 0$ with $\delta \le \varepsilon$. We shall now verify that this $\delta$ does the dastardly deed. To this end, pretend that $(x,y)$ be any pair of real numbers for which we have $0 < ((x-a)^2 + (y-b)^2)^{1/2} < \delta$. It then follows that

$$\begin{aligned} |g(x,y) - b| &= |y - b| = ((y-b)^2)^{1/2} \\ &\le ((x-a)^2 + (y-b)^2)^{1/2} < \delta \le \varepsilon. // \end{aligned}$$

6. (10 pts.) Let f(x,y) = tan(3x + 4y) and let $P_0 = (0,0)$.

(a) Find a unit vector in the direction in which f(x,y) increases most rapidly at $P_0$.

Since $\nabla f(x,y) = \left\langle 3\sec^2(3x+4y), 4\sec^2(3x+4y) \right\rangle$, the unit vector we want is the one in the same direction as that of the vector $\nabla f(0,0) = \langle 3, 4 \rangle$. Thus we want the unit vector

$$\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

(b) Compute the rate of change of f(x,y) at $P_0$ in the direction in which f(x,y) decreases most rapidly.

Since $\nabla f(0,0) = \langle 3, 4\rangle$, the rate we want is

$$-|\nabla f(0,0)| = -|\langle 3, 4 \rangle| = -5.$$

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7. (10 pts.) Obtain and locate the absolute extrema of the function $f(x,y) = x - y$ on the closed disk D with radius five centered at the origin. Observe that D is given in set builder notation as follows: $D = \{ (x,y) : x^2 + y^2 \leq 25 \}$. In performing this magic, use Lagrange multipliers to deal with the behavior of f on the boundary, $B = \{ (x,y) : x^2 + y^2 = 25 \}$.

// Since $f_x(x,y) = 1$ and $f_y(x,y) = -1$, f has no critical points. This means that all the extrema are located on the boundary, and yes, there are both flavors of extreme present since f is a continuous function defined on a closed and bounded region of the plane.

Set $g(x,y) = x^2 + y^2 - 25$. Then $(x,y)$ is on the circle defined by $x^2 + y^2 = 25$ precisely when $(x,y)$ satisfies $g(x,y) = 0$. Since $\nabla g(x,y) = \langle 2x, 2y\rangle$, $\nabla g(x,y) \neq \langle 0, 0\rangle$ when $(x,y)$ is on the circle given by $g(x,y) = 0$. Plainly f and g are smooth enough to satisfy the hypotheses of the Lagrange Multiplier Theorem. Thus, if a constrained local extremum occurs at $(x,y)$, there is a number $\lambda$ so that $\nabla f(x,y) = \lambda\nabla g(x,y)$. Now $\nabla f(x,y) = \lambda\nabla g(x,y) \Rightarrow \langle 1 , -1 \rangle = \lambda\langle 2x,2y\rangle \Rightarrow$ none of $x$, $y$, and $\lambda$ is zero and $x = -y$ by performing a little routine algebraic magic. Solving the system consisting of $x^2 + y^2 = 25$ and $x = -y$ yields $(5/2^{1/2},-5/2^{1/2})$, and $(-5/2^{1/2},5/2^{1/2})$. It turns out f is $5(2)^{1/2}$, the maximum value, at the first ordered pair, and f is $-5(2)^{1/2}$, the minimum value, at the second ordered pair. ["Slice a cylinder with a plane."]

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8. (10 pts.) Locate and classify the critical points of the function $f(x,y) = 8xy - 2x^2 - y^4$. Use the second partials test in making your classification. [Fill in the table below after you locate all the critical points.]

// Since $f_x(x,y) = 8y - 4x$ and $f_y(x,y) = 8x - 4y^3$, the critical points of f are given by the solutions to the following system:

$$8y - 4x = 0 \text{ and } 8x - 4y^3 = 0.$$

The system is plainly equivalent to $2y - x = 0$ and $2x - y^3 = 0$. Solving the system yields three critical points: (0,0), (4,2), and (-4,-2). [It’s easy. Just don’t lose any via division!!]

| Crit.Pt. | $f_{xx}$ @ c.p. | $f_{yy}$ @ c.p. | $f_{xy}$ @ c.p. | $\Delta$ @ c.p. | Conclusion |
|---|---|---|---|---|---|
| (x,y) | -4 | $-12y^2$ | 8 | $48y^2 - 64$ | |
| (0,0) | -4 | 0 | 8 | -64 | Saddle Point |
| (4,2) | -4 | -48 | 8 | 128 | Relative Maximum |
| (-4,-2) | -4 | -48 | 8 | 128 | Relative Maximum |

9. (10 pts.) Let $f(x,y) = \sin(x)\cos(y)$. Compute the gradient of f at $(\pi/3, -2\pi/3)$, and then use it to compute $D_{\mathbf{u}}f(\pi/3, -2\pi/3)$, where **u** is the unit vector in the same direction as **v** = <4,-3>.

$$\nabla f(x,y) = \langle \cos(x)\cos(y), -\sin(x)\sin(y)\rangle$$

$$\begin{aligned}\nabla f(\pi/3, -2\pi/3) &= \langle \cos(\tfrac{\pi}{3})\cos(-\tfrac{2\pi}{3}), -\sin(\tfrac{\pi}{3})\sin(-\tfrac{2\pi}{3})\rangle \\ &= \langle (\tfrac{1}{2})(-\tfrac{1}{2}), (-\tfrac{\sqrt{3}}{2})(-\tfrac{\sqrt{3}}{2})\rangle \\ &= \langle -\tfrac{1}{4}, \tfrac{3}{4}\rangle\end{aligned}$$

$$\boldsymbol{u} = \frac{1}{|\boldsymbol{v}|}\boldsymbol{v} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$

$$D_{\mathbf{u}}f(\pi/3,-2\pi/3) = \nabla f(\pi/3,-2\pi/3)\cdot\boldsymbol{u} = \left\langle -\frac{1}{4}, \frac{3}{4} \right\rangle\cdot\left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle = -\frac{13}{20}.$$

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10. (10 pts.) Recall that if $f(x,y)$ has continuous second order partial derivatives, then

$$\frac{\partial^2 f}{\partial x\,\partial y}(x,y) = \frac{\partial^2 f}{\partial y\,\partial x}(x,y).$$

Does there exist such a function f such that $f_x(x,y) = 2xy^3$ and $f_y(x,y) = 3x^2y^2 + 1$ ?? If the answer is "yes", obtain a formula for f.// Evidently

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(2xy^3) = 6xy^2 = \frac{\partial}{\partial x}(3x^2y^2+1) = f_{yx}(x,y) \textit{ for each } (x,y) \in \mathbb{R}^2.$$

Consequently, the answer here is "yes." [The domain is actually critical!! There will be more about this later in the course.]

Clearly

$$\frac{\partial f}{\partial x}(x,y) = 2xy^3 \Rightarrow f(x,y) = \int 2xy^3\,dx = x^2y^3 + c(y).$$

Therefore,

$$3x^2y^2 + 1 = \frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(x^2y^3) + \frac{dc}{dy}(y) \Rightarrow \frac{dc}{dy}(y) = 1 \Rightarrow c(y) = y + c_0$$

for some number $c_0$. Hence, $f(x,y) = x^2y^3 + y + c_0$.

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Silly 10 Point Bonus: Do at most one of the following problems. Indicate which unambiguously and say where your work is, for it won't fit here!

(a) Let g be defined by

$$g(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0). \end{cases}$$

Explicitly compute $g_{xy}(0,0)$ and $g_{yx}(0,0)$.

(b) Determine the absolute extrema of the function $f(x,y) = -y$ subject to the points $(x,y)$ lying on the curve defined by the equation $g(x,y) = 0$ where $g(x,y) = y^3 - x^2$. Guesses do not suffice here!