Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Vector objects must be denoted by using arrows. Do not "box" your final results. Show me all the magic on the page.

Silly 10 Point Bonus: Do at most one of the following problems. Indicate which unambiguously and say where your work is, for it won't fit here!

(a) Let g be defined by

$$g(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} , & (x,y)\neq(0,0) \\ 0 , & (x,y)=(0,0). \end{cases}$$

Explicitly compute $g_{xy}(0,0)$ and $g_{yx}(0,0)$.

(b) Determine the absolute extrema of the function $f(x,y) = -y$ subject to the points (x,y) lying on the curve defined by the equation $g(x,y) = 0$ where $g(x,y) = y^3 - x^2$. Guesses do not suffice here!

(a) To compute $g_{xy}(0,0)$ and $g_{yx}(0,0)$, due to the way that g is defined, we must resort to the definition of the two mixed partials as limits. Consequently, we will need to obtain $g_x(0,0)$, $g_y(0,0)$, and $g_x(x,y)$ and $g_y(x,y)$ for $(x,y) \neq (0,0)$ to achieve our goals.

First, to simplify our job, it helps to write g in a slightly easier to use form:

$$g(x,y) = \begin{cases} \dfrac{x^3y-xy^3}{x^2+y^2} , & (x,y)\neq(0,0) \\ 0 , & (x,y)=(0,0). \end{cases}$$

Then

$$g_x(0,0) = \lim_{h\to 0} \frac{g(0+h,0) - g(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0,$$

and

$$g_y(0,0) = \lim_{k\to 0} \frac{g(0,0+k) - g(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0.$$

Further, by doing several lines of routine algebra, one can obtain

$$g_x(x,y) = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2},$$

and

$$g_y(x,y) = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$$

for $(x,y) \neq (0,0)$.

Thus,

$$g_{xy}(0,0) = \lim_{k\to 0} \frac{g_x(0,0+k) - g_x(0,0)}{k} = \lim_{k\to 0} \frac{-(0+k)^5}{k(0+k)^4} = -1,$$

and

$$g_{yx}(0,0) = \lim_{h\to 0} \frac{g_y(0+h,0) - g_y(0,0)}{h} = \lim_{h\to 0} \frac{(0+h)^5}{h(0+h)^4} = 1. \quad //$$

(b) To determine the absolute extrema of the function

$$f(x,y) = -y$$

subject to the points (x,y) lying on the curve defined by the equation $g(x,y) = 0$ where $g(x,y) = y^3 - x^2$, it suffices to observe that the set of points in the plane where $g(x,y) = 0$ is the set

$$C = \{ (x,y) : y^3 - x^2 = 0 \}$$

$$= \{ (x,y) : y = x^{2/3} \}.$$

This means that f restricted to C is simply the function

$$h(x) = f(x,x^{2/3}) = -x^{2/3}$$

for $x \in \mathbb{R}$.

Since $h'(x) = -(2/3)x^{-1/3}$ for $x \neq 0$, it follows that h has a single critical point at $x = 0$, and that the First Derivative Test implies that h has a relative maximum there. Since this is the only critical point for h on $\mathbb{R}$, the relative maximum is a global or absolute maximum. Plainly, h has no absolute minimum value.

What this tells us is that f has an absolute maximum of 0 at (0,0) on the curve C and no minimum.//

Note: The Lagrange Multiplier Theorem is worthless here. If one attempts to use the multiplier equation, one ends up with two equations,

(1) $$0 = -2\lambda x$$

and

(2) $$-1 = 3\lambda y^2$$

which cannot be satisfied simultaneously by any number λ if (x,y) is on the curve C. This is not really a surprise. See Concepts: Questions and Discussion at the end of Section 9 that deals with Lagrange Multipliers. This provides an example for Question 3, for f has a maximum at (0,0) when restricted to C, but there is no number λ so that

$$\nabla f(0,0) = < 0, 1 > = \lambda < 0, 0 > = \lambda \nabla g(0,0).$$