Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Generic vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page neatly.

1. (10 pts.)
Let **F**(x,y,z) = $<xz^2,\ yx^2,\ zy^2>$. Compute the divergence and the curl of the vector field **F**.

(a)

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(yx^2) + \frac{\partial}{\partial z}(zy^2)$$

$$= z^2 + x^2 + y^2.$$

(b)

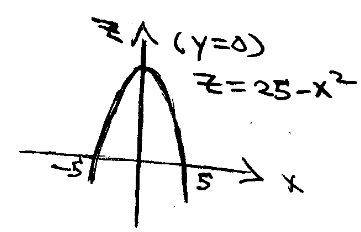
$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & yx^2 & zy^2 \end{vmatrix}$$

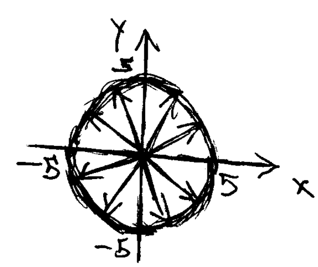
$$= < \frac{\partial}{\partial y}(zy^2) - \frac{\partial}{\partial z}(yx^2), -[\frac{\partial}{\partial x}(zy^2) - \frac{\partial}{\partial z}(xz^2)], \frac{\partial}{\partial x}(yx^2) - \frac{\partial}{\partial y}(xz^2) >$$

$$= < 2yz, 2xz, 2xy >.$$

2. (10 pts.) Compute the surface area of the part of the paraboloid defined by $z = 25 - x^2 - y^2$ that lies above the xy-plane. [Of course *above* means $z \geq 0$.]

$$\begin{aligned}
SA &= \iint_R \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 \right)^{1/2} dA \\
&= \iint_R ((-2x)^2 + (-2y)^2 + 1)^{1/2}\, dA \\
&= \iint_R (4x^2 + 4y^2 + 1)^{1/2}\, dA \\
&= \int_0^{2\pi}\int_0^5 (1 + 4r^2)^{1/2}\, r\, dr\, d\theta \\
&= \int_0^5 r(1 + 4r^2)^{1/2}\, dr \times \int_0^{2\pi} 1\, d\theta \\
&= 2\pi \int_1^{101} u^{1/2} \cdot \frac{1}{8}\, du \\
&= \frac{\pi}{4}\left(\frac{2}{3} u^{3/2}\right)\Big|_1^{101} = \frac{\pi}{6}((101)^{3/2} - (1)^{3/2}).
\end{aligned}$$

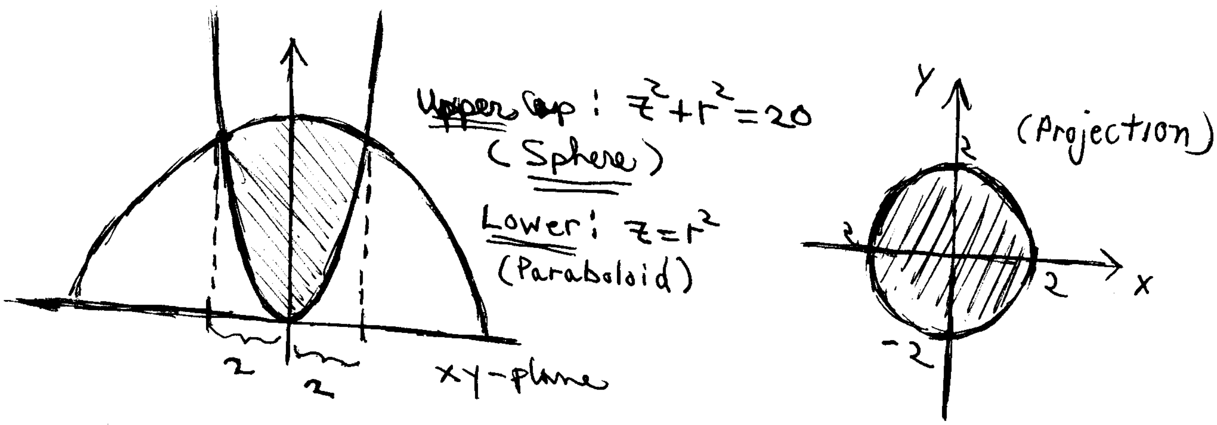




Obviously we have passed to polar coordinates along the way and then used the u-substitution $u = 4r^2 + 1$ to accomplish our goal.

3. (10 pts.) Write down the triple iterated integral in cylindrical coordinates that provides the numerical value of the volume of the region in 3-space bounded above by the surface $z = (20 - r^2)^{1/2}$ and below by the paraboloid $z = r^2$, but do not attempt to evaluate the integral you obtain.

$$\iiint_T 1\, dV = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{(20-r^2)^{1/2}} r\ dz\,dr\,d\theta.$$



By solving the system consisting of $z = (20 - r^2)^{1/2}$ and $z = r^2$, you can see that the projection on the xy-plane is the circle $r = 2$.

4. (10 pts.) Write down a triple iterated integral in cartesian coordinates that would be used to evaluate

$$\iiint_T f(x,y,z)\, dV\ ,$$

where $f(x,y,z) = x^2$ and T is the tetrahedron bounded by the coordinate planes and the part of the plane $x + y + z = 1$ that lies in the first octant, but do not attempt to evaluate the triple iterated integral you have obtained. [Sketching the traces in the coordinate planes will help.]

$$\iiint_T f(x,y,z)\, dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\, dz\,dy\,dx$$

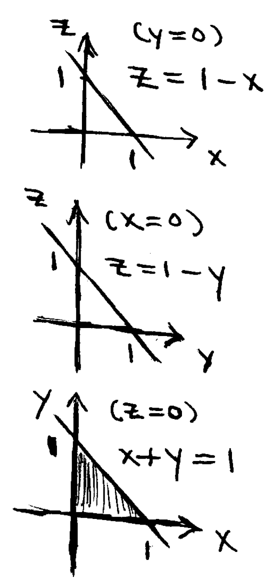
$$= \int_0^1\int_0^{1-y}\int_0^{1-x-y} x^2\, dz\,dx\,dy$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-z} x^2\, dy\,dz\,dx$$

$$= \int_0^1\int_0^{1-z}\int_0^{1-x-z} x^2\, dy\,dx\,dz$$

$$= \int_0^1\int_0^{1-y}\int_0^{1-y-z} x^2\, dx\,dz\,dy$$

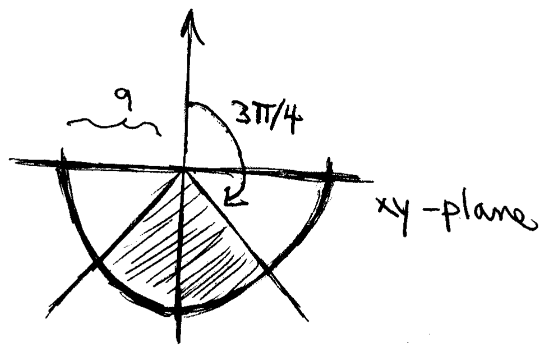
$$= \int_0^1\int_0^{1-z}\int_0^{1-y-z} x^2\, dx\,dy\,dz$$



Trifecta!! The tetrahedron is x-, y-, and z-simple.

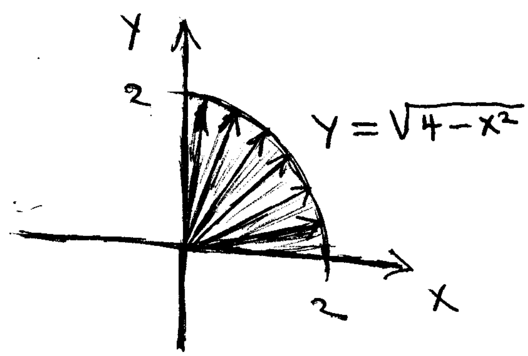
5. (10 pts.) Write down, but do not attempt to evaluate the triple iterated integral in spherical coordinates that provides the volume of the solid T that is bounded above by the cone $\phi = 3\pi/4$ and below by the sphere defined by $\rho = 9$.

$$\iiint_T 1\ dV = \int_0^{2\pi}\int_{3\pi/4}^{\pi}\int_0^9 \rho^2 \sin\phi\ d\rho d\phi d\theta.$$



6. (10 pts.) Convert the given iterated integral into an iterated integral in polar coordinates that has the same numerical value and is easier to evaluate, perhaps. Do not attempt to evaluate the polar integral. A picture might help.

$$\int_0^2\int_0^{(4-x^2)^{1/2}} (x^2+y^2)^{3/2}\ dydx = \int_0^{\pi/2}\int_0^2 (r^2)^{3/2} r\ drd\theta.$$



7. (10 pts.) Compute the value of the following line integral, where C is the path in the xy-plane from the point (-1,-1) to the point (1,1) along the curve $y = x^5$:

The ogre's parameterization for the curve, in the correct direction, is given in a vector form by $\mathbf{r}(t) = \langle t, t^5 \rangle$ for $t \in [-1,1]$. Consequently, $\mathbf{r}'(t) = \langle 1, 5t^4 \rangle$ and thus,

$$\int_C (y-x)\,dx + (yx^3)\,dy = \int_{-1}^{1} (t^5-t)(1) + (t^5t^3)(5t^4)\,dt$$

$$= \int_{-1}^{1} t^5 - t + 5t^{12}\,dt$$

$$= 10\int_0^1 t^{12}\,dt = \frac{10}{13}.$$

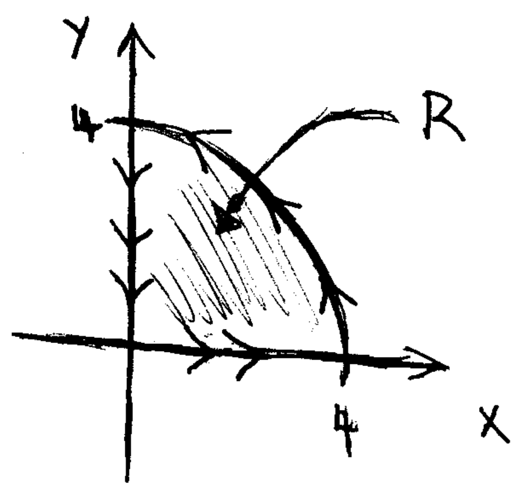
How did OgreOgre know that a parameterization is needed?? First, the diffential form is NOT EXACT. [Go check this, Frodo.] This means that the Fundamental Theorem of Line Integrals CANNOT BE USED. Second, the curve is NOT CLOSED, although the curve is simple. As a consequence, Green's Theorem CANNOT BE USED. This means finally, you are stuck with the "definition." A nice, easy parameterization is a must. This is actually an easy path integral, oddly. Or was it evenly????? [O.O. used both!!]

8. (10 pts.) Starting at the point (0,0), a particle goes along the x-axis until it reaches the point (4,0). It then goes from (4,0) to (0,4) along the circle with equation $x^2 + y^2 = 16$. Finally the particle returns to the origin by travelling along the y-axis. Use Green's Theorem to compute the work done on the particle by the force field defined by $\mathbf{F}(x,y) = \langle -5y, 5x \rangle$ for $(x,y) \in \mathbb{R}^2$. [Draw a picture. This is easy??]

$$W = \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C -5y\,dx + 5x\,dy$$

$$= \iint_R \frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(-5y)\,dA$$

$$= 10\iint_R 1\,dA = 10\cdot\text{Area(R)} = 40\pi.$$



If need be, you can compute the area of R via a variety of integrals. You can actually make this a difficult problem, at the end, if you don't pay attention.

9. (10 pts.) (a) Show that the vector field

$$\mathbf{F}(x,y) = < \cos(x)e^y + 10x,\ \sin(x)e^y >$$

is actually a gradient field by producing a function $\phi(x,y)$ such that $\nabla\phi(x,y) = \mathbf{F}(x,y)$ for all (x,y) in the plane.

Evidently

$$\frac{\partial}{\partial y}(\cos(x)e^y + 10x) = \cos(x)e^y = \frac{\partial}{\partial x}(\sin(x)e^y) \text{ for each } (x,y) \in \mathbb{R}^2.$$

Then

$$\frac{\partial\phi}{\partial x}(x,y) = \cos(x)e^y + 10x \Rightarrow \phi(x,y) = \int \cos(x)e^y + 10x\, dx$$

$$= \sin(x)e^y + 5x^2 + c(y).$$

Therefore,

$$\sin(x)e^y = \frac{\partial\phi}{\partial y}(x,y) = \frac{\partial}{\partial y}(\sin(x)e^y + 5x^2) + \frac{dc}{dy}(y) \Rightarrow \frac{dc}{dy}(y) = 0$$

$$\Rightarrow c(y) = c_0$$

for some number c_0. Hence, $\phi(x,y) = \sin(x)e^y + 5x^2 + c_0$.

(b) Using the Fundamental Theorem of Line Integrals, evaluate the path integral below, where C is any smooth path from the origin to the point $(\pi/2, \ln(2))$. [WARNING: You must use the theorem to get any credit here.]

$$\int_C (\cos(x)e^y + 10x)\, dx + (\sin(x)e^y)\, dy = \phi(x,y)\Big|_{(0,0)}^{(\pi/2,\ln(2))}$$

$$= \phi(\pi/2, \ln(2)) - \phi(0,0)$$

$$= \sin(\pi/2)e^{\ln(2)} + 5(\pi/2)^2$$

$$= 2 + \frac{5\pi^2}{4}.$$

10. (10 pts.) Use the substitution $u = x + y$ and $v = x - y$ to evaluate the integral below, where R is the bound region enclosed by the triangle with vertices at (0,0), (2,2), and (0,4).

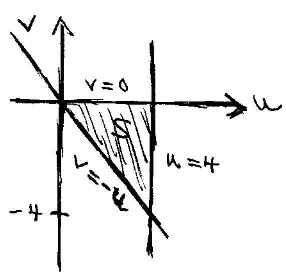
$$\iint_R x^2 - y^2 \, dx\,dy = \iint_R x^2 - y^2 \, dA_{x,y}$$

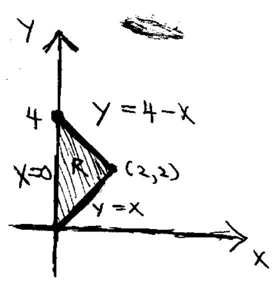
$$= \iint_S uv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{u,v}$$

$$= \int_0^4 \int_{-u}^0 uv \left| -\frac{1}{2} \right| dv\,du$$

$$= \frac{1}{2}\int_0^4 u \left[\int_{-u}^0 v \, dv \right] du$$

$$= \frac{1}{2}\int_0^4 -\frac{u^3}{2} \, dv = \left(\frac{-u^4}{16}\right)\Big|_0^4 = -\frac{4^4}{16} = -16.$$





$\mathbf{T}^{-1}$: $u = x+y$, $v = x-y$

$\mathbf{T}$: $x = (u+v)/2$,
$y = (u-v)/2$

Bounding Curves:
$y = x \Leftrightarrow v = 0$
$x = 0 \Leftrightarrow v = -u$
$y = 4 - x \Leftrightarrow u = 4$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = (1/2)(-1/2)-(1/2)(1/2) = -1/2.$$

Silly 10 Point Bonus: Reveal how one can obtain the exact value of the definite integral

$$\iint_R e^{-(x^2+y^2)} \, dA$$

where R = { (x,y): $x \geq 0$ and $y \geq 0$ }, even though there is no elementary anti-derivative for the function

$$f(x) = e^{-x^2}.$$

Say where your work is, for it won't fit here! [You may gloss some of the technical details related to the matter of convergence.]