Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Vector objects must be denoted by using arrows. Do not "box" your final results. Show me all the magic on the page.

1. (10 pts.) Suppose that an object is moving in a fixed plane with its acceleration given by **a**(t) = <0,-32>. Suppose that the initial position of the object is **r**(0) = <1,1> and the initial velocity of the object is **v**(0) = < -2, 2>.

(a) (6 pts.) Find the velocity of the object, **v**(t), as a function of time. // It follows easily from the vector-valued version of the Fundamental Theorem of Calculus that

$$\boldsymbol{v}(t) = \int_0^t \boldsymbol{a}(u)\,du + \boldsymbol{v}(0)$$

$$= \int_0^t \langle 0,-32\rangle\,du + \langle -2,2\rangle = \langle 0,-32t\rangle + \langle -2,2\rangle = \langle -2,2-32t\rangle.$$

(b) (2 pts.) Find the position of the object, **r**(t), as a function of time.
// Similarly,

$$\boldsymbol{r}(t) = \int_0^t \boldsymbol{v}(u)\,du + \boldsymbol{r}(0)$$

$$= \int_0^t \langle -2,2-32u\rangle\,du + \langle 1,1\rangle = \langle -2t,2t-16t^2\rangle + \langle 1,1\rangle$$

$$= \langle 1-2t,\ 1+2t-16t^2\rangle.$$

(c) (2 pts.) Obtain an equation for the parabola that is the path of the object.
// An equivalent set of parametric equations for **r**(t) is
$x = 1 - 2t$ and $y = 1 + 2t - 16t^2$. Solving for t in the "x" equation and substituting the result into the equation for "y" provides us with
$y = -2 + 7x - 4x^2$, an equation for the parabola.

2. (10 pts.) Let **r**(t) = < 4cos(t) , 4sin(t), 3t > for $t \in \mathbb{R}$. Then

(a) **r**′(t) = < (-4)sin(t) , (4)cos(t), 3 > ,

Note: ‖**r**′(t)‖ = 5

(b) **r**″(t) = < (-4)cos(t) , (-4)sin(t), 0 > ,

(c) **T**(t) = < (-4/5)sin(t) , (4/5)cos(t), 3/5 > ,

Note: ‖**T**′(t)‖ = 4/5

(d) **N**(t) = < -cos(t) , -sin(t), 0 > , and

(e) κ(t) = ‖**T**′(t)‖/‖**r**′(t)‖ = 4/25 .

3. (10 pts.) Obtain an arc-length parameterization for the curve **r**(t) = < t, 3cos(t), 3sin(t) > in terms of the initial point (0, 3, 0) which is the terminal point of **r**(0) in standard position. Rather than overloading the symbol **r**, write this new parameterization as **R**(s). How are **R** and **r** related?

First, since **r**(0) = <0, 3, 0>, the (signed) distance along the curve from the point given by **r**(0) on the graph to that given by **r**(t) is

$$\begin{aligned} s = \varphi(t) &= \int_0^t \|\mathbf{r}'(u)\|\, du \\ &= \int_0^t \|<1,-3\sin(u),3\cos(u)>\|\, du \\ &= \int_0^t 10^{1/2}\, du = 10^{1/2}(t-0) = 10^{1/2}t. \end{aligned}$$

Solving for t in terms of s yields

$$t = 10^{-1/2}s \quad \textit{so that } \varphi^{-1}(t) = 10^{-1/2}t.$$

Thus,

$$\begin{aligned} \mathbf{R}(s) &= \mathbf{r}(\varphi^{-1}(s)) = \mathbf{r}(10^{-1/2}s) \\ &= < \frac{s}{10^{1/2}}, 3\cos\left(\frac{s}{10^{1/2}}\right), 3\sin\left(\frac{s}{10^{1/2}}\right) >. \end{aligned}$$

Evidently, **R**(s) = **r**(φ^{-1}(s)), or equivalently, **r**(t) = **R**(φ(t)).

4. (10 pts.) A particle moves smoothly in such a way that at a particular time t = 0, we have **v**(0) = < 0 ,-4> and **a**(0) = < 2 , 3>. If we write **a**(0) in terms of **T**(0) and **N**(0), then

$$\mathbf{a}(0) = a_{\mathbf{T}}(0)\mathbf{T}(0) + a_{\mathbf{N}}(0)\mathbf{N}(0),$$

where

(a) **T**(0) = $(1/\|\mathbf{v}(0)\|)\mathbf{v}(0)$ = < 0 , -1 > ,

(b) $a_{\mathbf{T}}(0)$ = $\mathbf{a}(0)\cdot\mathbf{T}(0) = -3$,

(c) $a_{\mathbf{N}}(0)$ = $(\|\mathbf{a}(0)\|^2 - (a_{\mathbf{T}}(0))^2)^{1/2} = (13 - 9)^{1/2} = 2$, and

(d) **N**(0) = $(1/a_{\mathbf{N}}(0))(\mathbf{a}(0) - a_{\mathbf{T}}(0)\mathbf{T}(0))$ = < 1 , 0 > .

(e) Also, $\kappa(0) = \dfrac{\|\mathbf{v}(0)\times\mathbf{a}(0)\|}{\|\mathbf{v}(0)\|^3} = \dfrac{8}{64} = \dfrac{1}{8}$

since < 0, -4, 0 > X < 2, 3, 0 > = < 0, 0, 8 >.

5. (10 pts.) (a) Find the limit.

$$\lim_{t\to\infty}\left\langle \frac{t^2+1}{3t^2+2}, \frac{1}{t}\right\rangle = \left\langle \lim_{t\to\infty}\frac{t^2+1}{3t^2+2}, \lim_{t\to\infty}\frac{1}{t}\right\rangle = \left\langle \frac{1}{3}, 0\right\rangle$$

(b) Find parametric equations for the line tangent to the graph of $\mathbf{r}(t) = t^2\mathbf{i} + (2 - \ln(t))\mathbf{j}$ at the point where $t_0 = 1$.

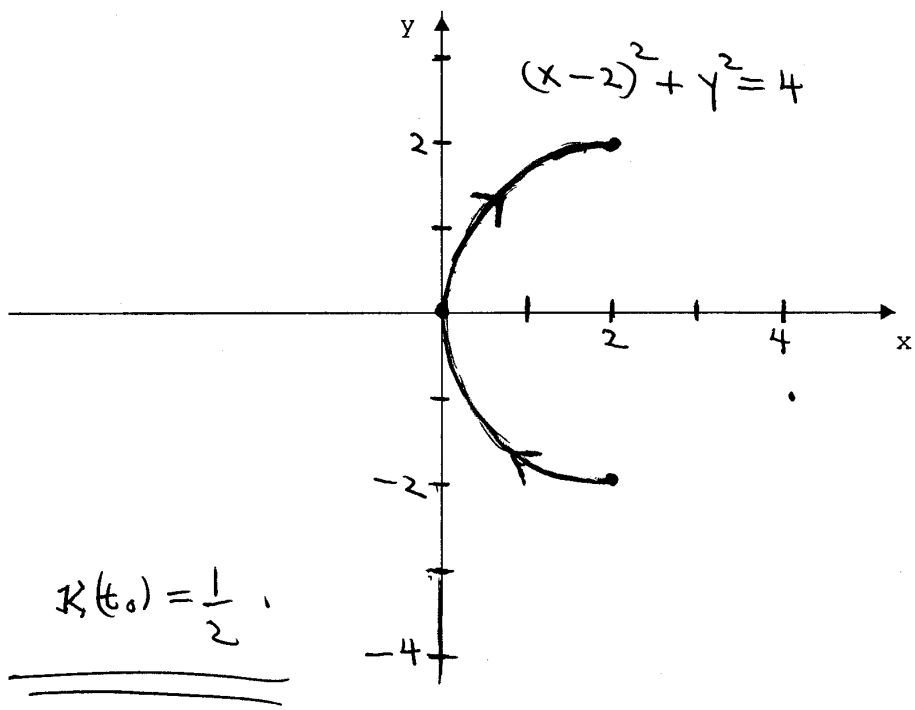
Since $\mathbf{r}'(t) = < 2t , -t^{-1} >$, $\mathbf{r}(1) = < 1 , 2 - \ln(1) > = < 1 , 2 >$, and $\mathbf{r}'(1) = < 2 , -1 >$. Consequently, an appropriate set of parametric equations is given by x = 1 + 2t and y = 2 - t.

6. (10 pts.) Let

$$\mathbf{r}(t) = < 2 + 2\cos(t), -2\sin(t) >$$

for $\pi/2 \le t \le 3\pi/2$. (a) Sketch the curve defined by this vector-valued function by eliminating the parameter t from the components x and y. Label very carefully. (b) Indicate the direction of increasing t. (c) Give the curvature $\kappa(t_0)$ of the curve at $\mathbf{r}(t_0)$ for each t_0. [Warning: Pythagoras is perched on your shoulder, but be careful of applying too much power.]

First observe that the graph of the vector-valued function is the same as that of the parametric equations $x = 2 + 2\cdot\cos(t)$ and $y = -2\cdot\sin(t)$ for t in the interval $[\pi/2, 3\pi/2]$. Consequently, if (x,y) lies on the curve, then (x,y) is a point on the graph of the circle defined by the equation $(x-2)^2 + y^2 = 4$, thanks to $\sin^2(t) + \cos^2(t) = 1$. By taking into account the domain of the function, we can obtain the graph. The curvature, of course, is the reciprocal of the radius. The osculating circle, Folks, is plainly the varmint defined by $(x-2)^2 + y^2 = 4$.



$\kappa(t_0) = \frac{1}{2}$.

7. (10 pts.) Let $f(x,y) = xy + 3$.

(a) $f(x + y\ ,\ x - y) = (x + y)(x - y) + 3 = x^2 - y^2 + 3$

(b) Find an equation of the level curve that passes through the point (-1,2).

Since $f(-1,2) = -2 + 3 = 1$, an equation for the level curve through (-1,2) is $1 = xy + 3$ or $xy = -2$, an equation a garden variety hyperbola.

8. (10 pts.). (a) Use limit laws and continuity properties to evaluate the following limit.

$$\lim_{(x,y)\to(1/2,\pi)} (xy^2\sin(xy)) = \frac{1}{2}(\pi)^2\sin(\frac{1}{2}\pi) = \frac{\pi^2}{2}$$

(b) Evaluate the limit, if it exists, by converting to polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{\tan(\pi(x^2+y^2))}{x^2+y^2} = \lim_{r\to0^+} \frac{\tan(\pi r^2)}{r^2}$$

$(L'H)$

$$= \lim_{r\to0^+}\frac{2\pi r\sec^2(\pi r^2)}{2r} = \pi$$

Silly 10 Point Bonus: To compute $g_{xy}(0,0)$ and $g_{yx}(0,0)$, due to the way that g is defined, we must resort to the definition of the two mixed partials as limits. Consequently, we will need to obtain $g_x(0,0)$, $g_y(0,0)$, and $g_x(x,y)$ and $g_y(x,y)$ for $(x,y) \neq (0,0)$ to achieve our goals.

First, to simplify our job, it helps to write g in a slightly easier to use form when $(x,y) \neq (0,0)$:

$$g(x,y) = \frac{x^3y - xy^3}{x^2+y^2}.$$

Then

$$g_x(0,0) = \lim_{h\to0} \frac{g(0+h,0) - g(0,0)}{h} = \lim_{h\to0} \frac{0}{h} = 0,$$

and

$$g_y(0,0) = \lim_{k\to0} \frac{g(0,0+k) - g(0,0)}{k} = \lim_{k\to0} \frac{0}{k} = 0.$$

Further, by doing several lines of routine algebra, one can obtain

$$g_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}, \text{ and } g_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \text{ for } (x,y) \neq (0,0).$$

Thus,

$$g_{xy}(0,0) = \lim_{k\to0} \frac{g_x(0,0+k) - g_x(0,0)}{k} = \lim_{k\to0} \frac{-(0+k)^5}{k(0+k)^4} = -1,$$

and

$$g_{yx}(0,0) = \lim_{h\to0} \frac{g_y(0+h,0) - g_y(0,0)}{h} = \lim_{h\to0} \frac{(0+h)^5}{h(0+h)^4} = 1. \quad //$$

9. (10 pts.) (a) Calculate $\partial z/\partial y$ using implicit differentiation when $(x^2 + y^2 + z^2)^{3/2} = 1$. Leave your answer in terms of x, y, and z.

By pretending z is a function of the two independent variables x and y and performing partial differentiations on both sides of the equation above, we have

$$\frac{\partial}{\partial y}\left[\left(x^2+y^2+z^2\right)^{3/2}\right] = \frac{\partial[1]}{\partial y} \Rightarrow \frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}\left(2y+2z\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$$

provided z is not zero.

(b) Find all second-order partial derivatives for the function $f(x,y) = x^2y^3$. Label correctly.

Since $f_x(x,y) = 2xy^3$ and $f_y(x,y) = 3x^2y^2$, we have $f_{xx}(x,y) = 2y^3$,

$f_{yy}(x,y) = 6x^2y$, and $f_{xy}(x,y) = f_{yx}(x,y) = 6xy^2$.

10. (10 pts.) (a) Compute the differential dz when

$$z = \tan^{-1}(xy).$$

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$= \frac{y}{1+(xy)^2}\,dx + \frac{x}{1+(xy)^2}\,dy.$$

(b) Let $f(x,y) = \ln(xy)$. Find the local linear approximation L to f at the point $P(1,2)$.

$$L(x,y) = f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)$$

$$= \ln(2)+(x-1)+\frac{1}{2}(y-2)$$

since

$$f_x(x,y) = \frac{1}{x} \text{ and } f_y(x,y) = \frac{1}{y}.$$

Silly 10 Point Bonus: Let g be defined by

$$g(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0). \end{cases}$$

Explicitly compute $g_{xy}(0,0)$ and $g_{yx}(0,0)$. [Say where your work is, for it won't fit here.] *This is on the bottom of Page 4 of 5.*