Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Generic vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page neatly.

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1. (10 pts.) Obtain an equation for the plane tangent to the graph of $f(x,y) = \sec(xy)$ when $(x_0,y_0) = (\pi,1/4)$. Then obtain a vector equation for the normal line to the surface at the same point. [**Reminder**: $z_0 = f(x_0,y_0)$]

First,

$$z_0 = f(\pi,\frac{1}{4}) = \sec(\frac{\pi}{4}) = \sqrt{2} \ .$$

Since

$$f_x(x,y) = y\sec(xy)\tan(xy) \quad \text{and} \quad f_y(x,y) = x\sec(xy)\tan(xy) \ ,$$

$$f_x(\pi,\frac{1}{4}) = \frac{1}{4}\sec(\frac{\pi}{4})\tan(\frac{\pi}{4}) = \frac{\sqrt{2}}{4} \quad \text{and} \quad f_y(\pi,\frac{1}{4}) = \pi\sec(\frac{\pi}{4})\tan(\frac{\pi}{4}) = \pi\sqrt{2} \ .$$

Consequently, an equation for the tangent plane at (x_0,y_0) is given by

$$z = \sqrt{2} + \frac{\sqrt{2}}{4}(x - \pi) + \pi\sqrt{2}(y - \frac{1}{4}) \ .$$

A vector equation for the normal line at (x_0,y_0) is

$$< x , y , z > = < \pi , \frac{1}{4} , \sqrt{2} > + t < \frac{\sqrt{2}}{4} , \pi\sqrt{2} , -1 > \ .$$

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2. (10 pts.) Let

$$f(x,y) = (y^2 - x^2)^{1/3}.$$

Compute the gradient of f at (-1,3). Then use it to compute the directional derivative $D_{\mathbf{u}}f(-1,3)$, where **u** is the unit vector forming an angle of $\theta = 5\pi/6$ with respect to the positive x-axis.

$$\nabla f(x,y) = \left\langle \frac{1}{3}(y^2-x^2)^{-2/3}(-2x) , \frac{1}{3}(y^2-x^2)^{-2/3}(2y) \right\rangle$$

$$\nabla f(-1,3) = \left\langle \frac{1}{6} , \frac{1}{2} \right\rangle$$

$$\boldsymbol{u} = \left\langle \cos(\frac{5\pi}{6}) , \sin(\frac{5\pi}{6}) \right\rangle = \left\langle -\frac{\sqrt{3}}{2} , \frac{1}{2} \right\rangle$$

$$D_{\mathbf{u}}f(-1,3) = \nabla f(-1,3)\cdot\boldsymbol{u} = \left\langle \frac{1}{6} , \frac{1}{2} \right\rangle\cdot\left\langle -\frac{\sqrt{3}}{2} , \frac{1}{2} \right\rangle = \frac{3 - \sqrt{3}}{12}$$

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3. (10 pts.) Suppose $f(x,y) = x^2y$. Find all unit vectors **u** so that

$$(*) \quad D_{\mathbf{u}}f(-1,2) = 0$$

is true for **u**.

First, $\nabla f(x,y) = \langle 2xy\,,\, x^2 \rangle \;\Rightarrow\; \nabla f(-1,2) = \langle -4\,,\, 1 \rangle.$

Thus,

$$\begin{aligned}(*) \;&\Leftrightarrow\; \nabla f(-1,2)\cdot\boldsymbol{u} = 0\\ &\Leftrightarrow\; -4u_1 + u_2 = 0\\ &\Leftrightarrow\; (**)\;\; u_2 = 4u_1\,,\ \textit{when}\ \boldsymbol{u} = \langle u_1, u_2\rangle.\end{aligned}$$

Since ***u*** must be a unit vector, ***u*** must also satisfy

$$(***) \quad u_1^2 + u_2^2 = \|\boldsymbol{u}\|^2 = 1.$$

Solving the system consisting of (**) and (***) above provides us with the desired unit vectors, namely

$$\boldsymbol{u}_1 = \left\langle \frac{1}{\sqrt{17}}\,,\,\frac{4}{\sqrt{17}} \right\rangle \quad \textit{and} \quad \boldsymbol{u}_2 = \left\langle -\frac{1}{\sqrt{17}}\,,\,-\frac{4}{\sqrt{17}} \right\rangle.$$

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4. (10 pts.) Assume $f(1,-2) = 4$ and $f(x,y)$ is differentiable at $(1,-2)$ with $f_x(1,-2) = 2$ and $f_y(1,-2) = -3$. Using an appropriate local linear approximation, estimate the value of $f(0.9,-1.950)$.

$$f(0.9,-1.950) \approx L(0.9,-1.950) = 3.65$$

This is Quick Check Exercise #4 of Section 14.4. Cheap thrills and an easy 10 points. You know, of course, we use the local linear approximation at (1,-2). All the pieces were handed to you on a platter with a cherry on top. You need only remember that the local linear approximation is given by

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,Y_0)(y-y_0)$$

and replace the pieces appropriately. Thus,

$$\begin{aligned}L(.9,-1.95) &= f(1,-2) + f_x(1,-2)(.9-1) + f_y(1,-2)(-1.95-(-2))\\ &= 4 + (2)(-.1) + (-3)(.05) = 4 - .2 - .15 = 3.65.\end{aligned}$$

5. (10 pts.) (a) Use an appropriate form of chain rule to find $\partial z/\partial v$ when $z = \cos(x)\sin(y)$ when $x = u - v$ and $y = u^2 + v^2$.

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\sin(x)\sin(y)(-1) + \cos(x)\cos(y)(2v)$$

$$= \sin(u-v)\sin(u^2+v^2) + \cos(u-v)\cos(u^2+v^2)(2v).$$

(b) Assume that $F(x,y,z) = 0$ defines z implicitly as a function of x and y. Show that if $\partial F/\partial z \neq 0$, then

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

Using classical curly d notation, since x and y are independent variables and z is a function of x and y, we have

$$\frac{\partial F(x,y,z)}{\partial y} = 0 \Rightarrow \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

6. (10 pts.) Locate all relative extrema and saddle points of the following function.

$$f(x,y) = x^4 + y^4 - 4xy$$

Use the second partials test in making your classification. (Fill in the table below after you locate all the critical points.)

Crit.Pt.	f_{xx} @ c.p.	f_{yy} @ c.p.	f_{xy} @ c.p.	D @ c.p.	Conclusion
(x,y)	$12x^2$	$12y^2$	-4	$144x^2y^2 - 16$	
(0,0)	0	0	-4	-16	Saddle Pt.
(1,1)	12	12	-4	128	Rel. Min
(-1,-1)	12	12	-4	128	Rel. Min

// Since $f_x(x,y) = 4x^3 - 4y$ and $f_y(x,y) = 4y^3 - 4x$, the critical points of f are given by the solutions to the following system: $4x^3 - 4y = 0$ and $4y^3 - 4x = 0$.

The system is plainly equivalent to $y = x^3$ and $x^9 - x = 0$. Solving the system yields three critical points: (0,0), (1,1), and (-1,-1). [Factor an x out of the two terms of the second equation and then repeatedly recognize differences of squares, Folks.] Then go up and fill in the silly table.

7. (10 pts.) (a) Evaluate the following iterated integral.

$$\int_0^{\ln(3)}\int_0^{\ln(2)} e^{x+y}\,dy\,dx = \int_0^{\ln(3)} e^x\left[\int_0^{\ln(2)} e^y\,dy\right]dx$$
$$= \left(\int_0^{\ln(2)} e^y\,dy\right)\cdot\left(\int_0^{\ln(3)} e^x\,dx\right)$$
$$= (2-1)\cdot(3-1) = 2.$$

(b) Very carefully write down, *but do not evaluate*, an iterated integral whose numerical value is the volume under the plane $z = 2x + y$ and over the rectangle $R = \{(x,y) : 3 \leq x \leq 5,\ 1 \leq y \leq 2\}$.

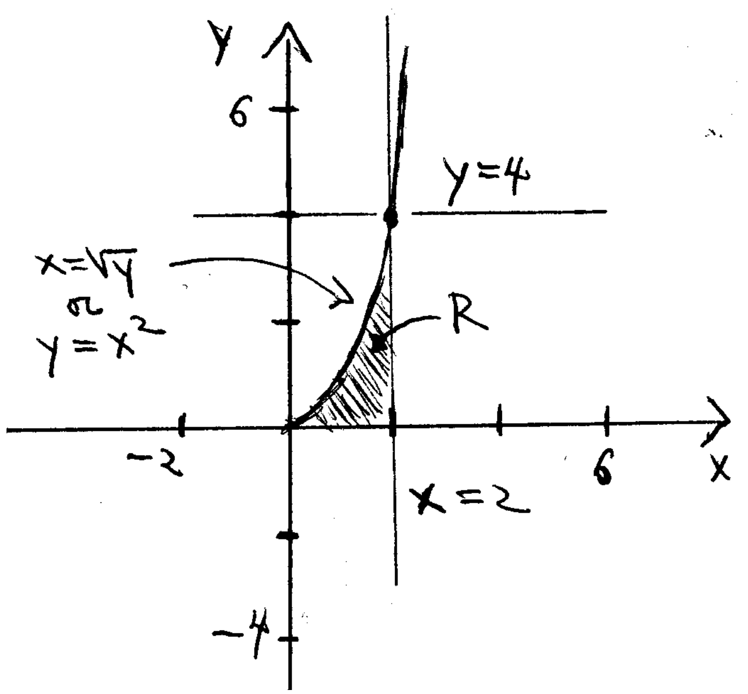
$$V = \int_3^5\int_1^2 2x+y\ dy\,dx$$

or

$$V = \int_1^2\int_3^5 2x+y\ dx\,dy.$$

8. (10 pts.) Evaluate the integral below by first reversing the order of integration. Sketching the region is a key piece of the puzzle.

$$\int_0^4\int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy = \int_0^2\int_0^{x^2} e^{x^3}\,dy\,dx$$
$$= \int_0^2 x^2 e^{x^3}\,dx$$
$$= \frac{1}{3}\int_0^8 e^u\,du$$
$$= \frac{1}{3}(e^8 - 1)$$



by using the u-substitution $u = x^3$ after reversing the order of integration.

The sketch to the right provides the information that makes this easy to do. Observe that we get the information to do the sketch from the limits of integration of the original iterated integral, namely that the region, R, consists of the pairs (x,y) which satisfy $y^{1/2} \leq x \leq 2$ when $0 \leq y \leq 4$. The bounding curves may be read from these inequalities easily.

9. (15 pts.) Let $f(x,y) = x^2 - y^2$ on the closed unit disk defined by $x^2 + y^2 \leq 1$. Find the absolute extrema and where they occur. Use Lagrange multipliers to analyze the function on the boundary.

// First, we deal with interior points of the disk. Since $\nabla f(x,y) = \langle -2x, 2y \rangle$, the only critical point is $(x,y) = (0,0)$. Obviously, $f(0,0) = 0$.

Now to study f on the boundary using Lagrange multipliers, set $g(x,y) = x^2 + y^2 - 1$. Then (x,y) is on the circle defined by $x^2 + y^2 = 1$ precisely when (x,y) satisfies $g(x,y) = 0$. Since $\nabla g(x,y) = \langle 2x, 2y \rangle$, $\nabla g(x,y) \neq \langle 0, 0 \rangle$ when (x,y) is on the circle given by $g(x,y) = 0$. Plainly f and g are smooth enough to satisfy the hypotheses of the Lagrange Multiplier Theorem. Thus, if a constrained local extremum occurs at (x,y), there is a number λ so that $\nabla f(x,y) = \lambda \nabla g(x,y)$. Now

$$\nabla f(x,y) = \lambda \nabla g(x,y) \Rightarrow \langle -2x, 2y \rangle = \lambda \langle 2x, 2y \rangle \Rightarrow 0 = 4xy$$

$$\Rightarrow x = 0 \text{ or } y = 0$$

by performing a little routine algebraic magic. Solving each of the systems consisting of

(a) $x^2 + y^2 = 1$ and $x = 0$,

and

(b) $x^2 + y^2 = 1$ and $y = 0$,

yields the desired (constrained) critical points, (0,1), (0,-1), (1,0), and (-1,0). It turns out $f(0,1) = f(0,-1) = 1$ and $f(1,0) = f(-1,0) = -1$. So -1 is the minimum and occurs at (1,0) and (-1,0), and 1 is the maximum and occurs at the two points (0,1) and (0,-1). A deeper analysis would show we have a saddle point at (0,0). Since the Extreme-value Theorem ensures we have absolute extrema, we only need to know the function value there.//

10. (5 pts.) Find a unit vector in the direction in which $f(x,y)$ increases most rapidly at $P_0 = (0,\pi/6)$ when

$$f(x,y) = \cos(3x - y).$$

Since

$$\nabla f(x,y) = \langle -\sin(3x-y)(3), -\sin(3x-y)(-1) \rangle ,$$

we have

$$\nabla f(0,\frac{\pi}{6}) = \left\langle -3\sin(-\frac{\pi}{6}), \sin(-\frac{\pi}{6}) \right\rangle = \left\langle \frac{3}{2}, -\frac{1}{2} \right\rangle.$$

Thus, a unit vector providing the direction of maximum increase at the desired point is

$$\boldsymbol{u} = \frac{1}{\|\nabla f(0,\frac{\pi}{6})\|} \nabla f(0,\frac{\pi}{6}) = \left\langle \frac{3}{10^{1/2}}, -\frac{1}{10^{1/2}} \right\rangle.$$

Silly 10 Point Bonus: (a) State the definition of differentiability for a function of two variables. [You may either state the definition found in the text or the one given by the instructor in class.] (b) Then using only the definition you stated, show the function

$$f(x,y) = xy$$

is differentiable at any point (x,y) in the plane. Say where your work is, for it won't fit here. Answers found in c3-t3-bo.pdf.